

TANNAKA DUALITY AND STABLE INFINITY-CATEGORIES

ISAMU IWANARI

1. INTRODUCTION

The theory of *tannakian categories* due to Grothendieck-Saavedra [36], Deligne-Milne [10], Deligne [8], [9] says that the symmetric monoidal abelian categories of representations of a pro-algebraic group is characterized as a symmetric monoidal abelian category which satisfies some categorical conditions. Its characterization is interesting in its own right. Beside, it has many applications; notably, it allows one to obtain pro-algebraic groups from various categories, which encode the data of categories as their representations (e.g. Picard-Vessiot theory, Nori's fundamental group schemes). Similarly, the theory of *Galois categories* [15] by Grothendieck characterizes Cartesian symmetric monoidal categories of representations of pro-finite groups. Let us reformulate slightly the category of representations. If G is a pro-algebraic group, then any representation of G corresponds to a quasi-coherent sheaf on the classifying stack BG . Namely, the symmetric monoidal category of quasi-coherent sheaves on BG may be viewed as that of representations of G . With this in mind, we can summarize the situation by saying that a tannakian theory provides a correspondence between geometric objects (e.g. BG) and symmetric monoidal categories which satisfy some condition.

Our main results of this paper may be best understood as tannakian results. Let us shift our interest to the world of higher category theory. The purpose of this paper is to establish tannakian results for *symmetric monoidal stable ∞ -categories* [31] with coefficients in a field of *characteristic zero*. In a sense, stable ∞ -categories can be considered as a correct generalization of triangulated categories in the realm of ∞ -categories (cf. e.g. [30], [31], [4]), and in the present paper our interest lies in stable ∞ -categories.

Our principal result of this paper is a tannakian characterization. We introduce the notion of *fine ∞ -categories* (or *fine tannakian ∞ -categories*). Let k be a field of characteristic zero. Let \mathcal{C}^\otimes be a k -linear symmetric monoidal stable idempotent complete ∞ -category.

Definition 1.1. Let C be an object of \mathcal{C} . We say that C is *wedge-finite* (or *exterior-finite*) if there is a natural number $n \geq 0$ such that $\wedge^{n+1}C \simeq 0$ and $\wedge^n C$ is invertible in \mathcal{C}^\otimes . We call n the dimension of C . Here an object C of \mathcal{C} is said to be invertible if there is an object C' such that $C \otimes C' \simeq C' \otimes C \simeq 1_{\mathcal{C}}$ where $1_{\mathcal{C}}$ is a unit of \mathcal{C}^\otimes . The n -fold wedge product $\wedge^n C$ is defined to be the image of the idempotent map $\text{Alt}^n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \text{sign}(\sigma) \sigma : C^{\otimes n} \rightarrow C^{\otimes n}$, i.e. $\text{Ker}(1 - \text{Alt}^n)$, in the homotopy category $\text{h}(\mathcal{C})$ that is an idempotent complete triangulated category. The symmetric group Σ_n acts on $C^{\otimes n}$ by permutation. By convention a zero object is a 0-dimensional wedge-finite object.

Remark 1.2. By definition the notion of wedge-finiteness descends to the level of the homotopy category. Thus one can check this condition at the level of symmetric monoidal triangulated categories. Any symmetric monoidal functor preserves wedge-finite objects. If the endomorphism algebra $\text{End}_{\text{h}(\mathcal{C})}(1_{\mathcal{C}})$ of a unit object in the homotopy category is a field, the invertibility of $\wedge^n C$ in Definition 1.1 is automatic (see Proposition 6.1).

Definition 1.3. Let \mathcal{C}^\otimes be a k -linear symmetric monoidal stable presentable ∞ -category. We say that \mathcal{C}^\otimes is a *fine ∞ -category* over k (or *fine tannakian ∞ -category*) if

- (i) there is a small set $\{C_\alpha\}_{\alpha \in A}$ of (dualizable) wedge-finite objects such that \mathcal{C}^\otimes is generated by $\{C_\alpha, C_\alpha^\vee\}_{\alpha \in A}$ as a symmetric monoidal stable presentable ∞ -category (cf. Definition 1.8). Here C_α^\vee denotes the dual of C_α .
- (ii) a unit object is compact (cf. [30, 5.3.5], Remark 1.9).

We refer to $\{C_\alpha\}_{\alpha \in A}$ having the property (i) as a set of wedge-finite (or exterior-finite) generators. Here “fine” may be considered as an abbreviation for “finiteness” + “exterior-product”. If no confusion seems likely to arise, we usually omit “over k ”.

Our characterization theorem is the following (cf. Theorem 4.1, Theorem 4.5):

Theorem 1.4 (Characterization theorem). *Let \mathcal{C}^\otimes be a k -linear symmetric monoidal stable presentable ∞ -category. The followings are equivalent to one another:*

- (1) \mathcal{C}^\otimes is a fine ∞ -category.
- (2) *There exist a derived quotient stack $X = [\mathrm{Spec} A/G]$ where a pro-reductive group G acts on an affine derived scheme $\mathrm{Spec} A$ with A a commutative differential graded algebra, and a k -linear symmetric monoidal equivalence $\mathcal{C}^\otimes \simeq \mathrm{QC}^\otimes(X)$. Here $\mathrm{QC}^\otimes(X)$ denotes the symmetric monoidal stable ∞ -category of quasi-coherent complexes on X (see Section 2.3).*

A derived stack is a stack in the theory of derived algebraic geometry. Derived algebraic geometry is a generalization of classical algebraic geometry [32], [41] which brings homotopy-theoretic ideas and techniques. We here think of derived stacks of the form $[\mathrm{Spec} A/G]$ appeared in Theorem 1.4 as the generalization of classifying stacks of affine group schemes as well as nice class of derived stacks. This characterization makes it possible to obtain a derived stack $X = [\mathrm{Spec} A/G]$ from an abstract symmetric monoidal stable ∞ -category. Our interest does not restrict to the gerbe-like stacks but we consider the class of nice derived quotient stacks, therefore one gains access to the extensive power of derived algebraic geometry. More importantly, our construction of a derived quotient stack (from a fine ∞ -category with a given set of wedge-finite generators) is quite explicit, and the associated stack has a specific form; see Section 4.

Fine ∞ -categories are defined by reasonably simple conditions. By verifying conditions we can find examples of fine ∞ -categories in practice. Recent fascinating development of higher category theory has attracted our attention to various examples of symmetric monoidal stable ∞ -categories. Among them, the next result proves that the followings are fine ∞ -categories (see Section 6 for details):

Theorem 1.5. *The followings are examples of fine ∞ -categories:*

- (i) *the unbounded derived ∞ -category of representations of a pro-algebraic algebraic group over a field of characteristic zero,*
- (ii) *the stable ∞ -category of mixed motives generated by Kimura finite dimensional Chow motives,*
- (iii) *the stable ∞ -category of noncommutative mixed motives generated by Kimura finite dimensional noncommutative motives,*
- (iv) *the stable ∞ -category of Ind-coherent complexes on a topological space,*
- (v) *the unbounded derived ∞ -category of quasi-coherent complexes on a quasi-projective variety.*

The examples (ii) and (iii) are of great interest in view of motivic Galois theory of mixed motives. The striking aspect of (ii) is that it reveals an intimate relation between Kimura finiteness of motives and Theorem 1.4 about fine ∞ -categories (see Section 6.3). The example (iv) is related with rational homotopy theory (see Section 6.5). We will prove that the associated stack encodes the rational homotopy type of a topological space. As an illustration, we briefly recall two examples of Galois categories: the category of finite covers of a topological space and the category of finite étale covers of a field. Namely, the theory of Galois categories

simultaneously generalizes the fundamental groups of topological spaces and the classical Galois theory:

$$(\text{Classical Galois theory}) \leftarrow (\text{Galois categories}) \rightarrow (\pi_1 \text{ of topological spaces}).$$

Motivic Galois theory generalizes the classical Galois theory to motives generated by general algebraic varieties, and the rational homotopy theory is a generalization of fundamental groups up to torsion. Hence one may think of examples (ii) and (iv) of fine ∞ -categories as a “higher” generalization of the above picture.

The main difficulty in the proof of Theorem 1.4 arises from the fact that $\mathrm{QC}^\otimes([\mathrm{Spec} A/G])$ (or a given symmetric monoidal stable ∞ -category) does not have a tannakian category or the like as its full subcategory in general, so that in our setting it is hard to rely on the classical tannakian theory and methods. We use a new way of characterizing the derived ∞ -category of representations of a general linear group GL_d by a universal property. It may be of independent interest but is a key ingredient to the proof of Theorem 1.4 (cf. Theorem 3.1):

Theorem 1.6 (A universal property). *Let \mathcal{C}^\otimes be a k -linear symmetric monoidal stable pre-sentable ∞ -category whose tensor product preserves colimits separately in each variable. Let $\mathcal{C}_{\wedge,d}$ be the full subcategory of d -dimensional wedge-finite (exterior-finite) objects in \mathcal{C}^\otimes and $\mathcal{C}_{\wedge,d}^\simeq$ the largest Kan subcomplex (i.e., ∞ -groupoid) of the underlying ∞ -category $\mathcal{C}_{\wedge,d}^\simeq$. Then there exists a natural homotopy equivalence of spaces*

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})}(\mathrm{QC}^\otimes(\mathrm{BGL}_d), \mathcal{C}^\otimes) \rightarrow \mathcal{C}_{\wedge,d}^\simeq$$

which carries $f : \mathrm{QC}^\otimes(\mathrm{BGL}_d) \rightarrow \mathcal{C}^\otimes$ to the image $f(K)$ of the standard representation K of GL_d . That is, an object $C \in \mathcal{C}_{\wedge,d}$ corresponds to a k -linear symmetric monoidal functor $\mathrm{QC}^\otimes(\mathrm{BGL}_d) \rightarrow \mathcal{C}^\otimes$ that sends K to C .

The classical tannakian theory tells us that for a pro-algebraic group G over k and a k -algebra R , the groupoid $\mathrm{Map}_{k\text{-stacks}}(\mathrm{Spec} R, BG)$ of morphisms to BG is naturally equivalent to the groupoid $\mathrm{Map}_k^\otimes(\mathrm{qcoh}^\otimes(BG), \mathrm{qcoh}^\otimes(\mathrm{Spec} R))$ of k -linear symmetric monoidal exact functors between symmetric monoidal abelian categories of quasi-coherent sheaves; $f : \mathrm{Spec} R \rightarrow BG$ corresponds to $f^* : \mathrm{qcoh}^\otimes(BG) \rightarrow \mathrm{qcoh}^\otimes(\mathrm{Spec} R)$, cf. [10] for precise details. Its analogue for derived ∞ -categories of schemes and Deligne-Mumford stacks is proved in [14]. We now invite the reader’s attention to the fact that in the setting of our derived (Artin) stacks symmetric monoidal functors do *not* correspond to morphisms of stacks. There exists a symmetric monoidal functor which is not the pullback functor of a morphism of stacks: Let $B\mathbb{G}_m$ is the usual classifying stack of the algebraic torus \mathbb{G}_m . We have a symmetric monoidal equivalence

$$\mathrm{QC}^\otimes(B\mathbb{G}_m) \rightarrow \mathrm{QC}^\otimes(B\mathbb{G}_m)$$

which carries each character χ_n of weight n of \mathbb{G}_m to $\chi_n[2n]$. But it does not arise as the pullback functor of any morphism $B\mathbb{G}_m \rightarrow B\mathbb{G}_m$ (because it does not preserve its heart of standard t -structure). To analyze this exotic and new phenomenon¹, inspired by [14] we introduce the geometric notion of *correspondences* between derived stacks. A correspondence from X to Y can be viewed as a twisted morphism and is defined in a similar way to algebraic correspondences. This notion captures the phenomenon. That is, we prove that correspondences (rather than morphisms) corresponds to symmetric monoidal functors (see Section 5):

Theorem 1.7 (Symmetric monoidal functors versus correspondences). *Let $X = [\mathrm{Spec} A/G]$ and $Y = [\mathrm{Spec} B/H]$ be two quotient stacks where $A, B \in \mathrm{CAlg}_k$, and G and H are pro-reductive groups over k . There is a natural equivalence of ∞ -groupoids*

$$\mathrm{Map}_{\mathrm{Cor}_k}(X, Y) \rightarrow \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})}(\mathrm{QC}^\otimes(Y), \mathrm{QC}^\otimes(X)); f \mapsto f^*.$$

¹But such phenomena naturally appear in some generalizations of tannaka duality. For example, it appears in Tannaka duality for (braided) monoidal categories, which involves Drinfeld associator and twist.

Here the left hand side is the spaces of correspondences from X to Y (defined in Section 5). Moreover, the composition of symmetric monoidal functors corresponds to a composition of correspondences.

Let us remember that there are two aspects of tannakian theory of a given symmetric monoidal category \mathcal{C}^\otimes . One is to think of \mathcal{C}^\otimes as the category of sheaves on a geometric object (or the representation category of a group object). The other is to consider the group object which represents automorphism group of p when \mathcal{C}^\otimes equips with a “fiber functor” p . Let us focus on the second aspect. Suppose that a symmetric monoidal stable ∞ -category \mathcal{C}^\otimes equips with a symmetric monoidal functor $p : \mathcal{C}^\otimes \rightarrow \text{Mod}_k^\otimes$ to a symmetric monoidal stable ∞ -category of Hk -module spectra. In [24] we have constructed a derived affine group scheme which represents the automorphism group $\text{Aut}(p)$ of p . We refer to [24], [25] for details. When \mathcal{C}^\otimes is a fine ∞ -category (and thus $\mathcal{C}^\otimes \simeq \text{QC}^\otimes([\text{Spec } A/G])$), one can apply the construction of a based loop space for $[\text{Spec } A/G]$ (i.e., G -equivariant bar construction), under a suitable condition, to obtain a derived affine group scheme $G := \Omega_*[\text{Spec } A/G]$ which represents the automorphism group of p (see Remark 4.13 and 6.13, Section 6.5). This derived group scheme G is the Tannaka dual of \mathcal{C}^\otimes with respect to p .

We would like to mention our subsequent work. The results in this paper have found applications to mixed motives. We apply main results of this paper to the study of a motivic Galois group for mixed motives tensor-generated by an abelian variety. In particular, we prove a conjectural structure of the motivic Galois group of such motives (see [26]). Our tannakian results of this paper can be applied also to motivic unipotent fundamental groups of curves, the universal family of a modular variety, etc., and thus one may expect more to this and other directions.

Next we would like to recall a recent progress on tannakian theory for symmetric monoidal stable ∞ -categories *endowed with t -structures*. Lurie [32, VIII Section 4] establishes a tannakian theory of symmetric monoidal stable ∞ -categories with coefficients in a field of characteristic zero which are endowed with t -structures and satisfy some conditions (locally dimensional ∞ -categories), and in [46] a version of tannakian theory for stable ∞ -categories over ring spectra equipped with t -structures and fiber functors is developed. As well as the motivation from motives, Deligne’s idea [8], [9] and Lurie’s idea on beautiful internal characterizations of tannakian (and super-tannakian, locally dimensional) categories without fiber functors influence our work. Meanwhile, as we can easily imagine, there are substantial differences between the present paper and theories taking account of t -structures. Firstly, if a symmetric stable ∞ -category is endowed with t -structure, its heart is a tannakian category (or a suitable symmetric monoidal abelian category) under an appropriate condition on t -structure. Thus unlike the setting of this paper, one can rely on the classical theory of tannakian category or a similar argument. Secondly, since we do not assume t -structures, thus Theorem 1.4 is relatively easy to apply. For example, it is crucial to unconditional applications to mixed motives (cf. [26], Section 6). Thirdly, as observed above, symmetric monoidal functors of fine ∞ -categories correspond to not morphisms of derived stacks but correspondences.

This paper is organized as follows. In Section 2 we recall/prepare basic definitions and results about derived stacks, symmetric monoidal stable ∞ -categories, and quasi-coherent complexes, etc. In Section 3, we discuss a universal characterization of the derived ∞ -category of representations of a general linear group in terms of wedge-finite objects. Namely we prove Theorem 1.6. In Section 4, we prove Theorem 1.4 and its algebraic version Theorem 4.1. Moreover, we study an explicit presentation of the derived stack associated to a fine ∞ -category (together with a prescribed wedge-finite generator). In Section 5, we introduce correspondences between derived stacks and prove Theorem 1.7. The reader can skip this Section for the first reading. In Section 6, we present some examples of fine ∞ -categories. We discuss (i) the relation with

the classical tannakian categories, (ii) unconditional application to stable ∞ -category of mixed motives, which opens up a nice relationship with Kimura finiteness of motives, (iii) derived ∞ -category of quasi-coherent sheaves on a quasi-projective variety, (iv) a tannakian theory between coherent sheaves on a topological space and rational homotopy theory. The author is partially supported by Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science. I would like to thank Prof. K. Fujiwara, K. Kimura, S. Kimura, S. Mochizuki, S. Moriya and T. Yamazaki for valuable conversations related to the subject of this paper.

Convention and notation. Throughout this paper we use the theory of *quasi-categories*. A quasi-category is a simplicial set which satisfies the weak Kan condition of Boardman-Vogt. The theory of quasi-categories from the viewpoint of higher category theory were extensively developed by Joyal and Lurie [27], [30], [31]. Following [30] we shall refer to quasi-categories as ∞ -categories. Our main references are [30] and [31]. For the brief introduction to ∞ -categories, we refer to [30, Chapter 1], [16], [14, Section 2]. For the quick survey on various approaches to $(\infty, 1)$ -categories (e.g. simplicial categories, Segal categories, complete Segal spaces, etc) and their relations, we refer to [4]. As a set-theoretic foundation, we employ the axiom of ZFC together with the axiom of Grothendieck universes (i.e., every Grothendieck universe is an element of a larger universe). We fix a sequence of universes $(\mathbb{N} \in) \mathbb{U} \in \mathbb{V} \in \mathbb{W} \in \dots$ and refer to sets belonging to \mathbb{U} (resp. \mathbb{V} , \mathbb{W}) to as small sets (resp. large sets, super-large sets). But in the text we avoid using the notation \mathbb{U} , \mathbb{V} , \mathbb{W} . To an ordinary category we can assign an ∞ -category by taking its nerve, and therefore when we treat ordinary categories we often omit the nerve $N(-)$ and directly regard them as ∞ -categories. We often refer to a map $S \rightarrow T$ of ∞ -categories as a functor. We call a vertex in an ∞ -category S (resp. an edge) an object (resp. a morphism). Here is a list of (some) of the convention and notation that we will use:

- Δ : the category of linearly ordered finite sets (consisting of $[0], [1], \dots, [n] = \{0, \dots, n\}, \dots$)
- Δ^n : the standard n -simplex
- N : the simplicial nerve functor (cf. [30, 1.1.5])
- \mathcal{C}^{op} : the opposite ∞ -category of an ∞ -category \mathcal{C}
- Let \mathcal{C} be an ∞ -category and suppose that we are given an object c . Then $\mathcal{C}_{c/}$ and $\mathcal{C}_{/c}$ denote the undercategory and overcategory respectively (cf. [30, 1.2.9]).
- \mathcal{C}^{\simeq} : the largest Kan subcomplex (contained) in an ∞ -category \mathcal{C} , that is, the Kan complex obtained from \mathcal{C} by restricting morphisms (edges) to equivalences.
- Cat_{∞} : the ∞ -category of small ∞ -categories
- $\widehat{\text{Cat}}_{\infty}$: ∞ -category of large ∞ -categories
- \mathcal{S} : ∞ -category of small spaces. We denote by $\widehat{\mathcal{S}}$ the ∞ -category of large ∞ -spaces (cf. [30, 1.2.16])
- $\mathbf{h}(\mathcal{C})$: homotopy category of an ∞ -category (cf. [30, 1.2.3.1])
- $\text{Ind}(\mathcal{C})$: ∞ -category of Ind-objects in an ∞ -category \mathcal{C} (see [30, 5.3.5.1], [31, 4.8.1.13] for the symmetric monoidal setting).
- $\text{Fun}(A, B)$: the function complex for simplicial sets A and B
- $\text{Fun}_{\mathcal{C}}(A, B)$: the simplicial subset of $\text{Fun}(A, B)$ classifying maps which are compatible with given projections $A \rightarrow C$ and $B \rightarrow C$.
- $\text{Map}(A, B)$: the largest Kan subcomplex of $\text{Fun}(A, B)$ when A and B are ∞ -categories,
- $\text{Map}_{\mathcal{C}}(C, C')$: the mapping space from an object $C \in \mathcal{C}$ to $C' \in \mathcal{C}$ where \mathcal{C} is an ∞ -category. We usually view it as an object in \mathcal{S} (cf. [30, 1.2.2]).

Stable ∞ -categories, symmetric monoidal ∞ -categories and spectra. For the definitions of (symmetric) monoidal ∞ -categories and ∞ -operads, their algebra objects, we shall refer to [31]. A stable ∞ -category is an ∞ -category which satisfies the conditions (i) there is a zero object, i.e., an object which is both initial and final, (ii) every morphism has a fiber and a cofiber, (iii) for any sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms, X is a fiber of g if and only if Z is a cofiber of

f (see [31, 1.1.1.9]). Our reference for stable ∞ -categories is [31, Chapter 1]. We list some of notation.

- Mod_A : ∞ -category of A -module spectra for a commutative ring spectrum A . When R is the Eilenberg-MacLane spectrum HK of an ordinary commutative ring K , we write Mod_K for Mod_R (thus Mod_K is not the category of usual K -modules).
- Fin_* : the category of pointed finite sets $\langle 0 \rangle = \{*\}$, $\langle 1 \rangle = \{1, *\}$, \dots , $\langle n \rangle = \{1, \dots, n, *\}$, \dots . A morphism is a map $f : \langle n \rangle \rightarrow \langle m \rangle$ such that $f(*) = *$. Note that f is not assumed to be order-preserving.
- Let $\mathcal{M}^\otimes \rightarrow \mathcal{O}^\otimes$ be a fibration of ∞ -operads. We denote by $\text{Alg}_{/\mathcal{O}^\otimes}(\mathcal{M}^\otimes)$ the ∞ -category of algebra objects (cf. [31, 2.1.3.1]). We often write $\text{Alg}(\mathcal{M}^\otimes)$ or $\text{Alg}(\mathcal{M})$ for $\text{Alg}_{/\mathcal{O}^\otimes}(\mathcal{M}^\otimes)$. Suppose that $\mathcal{P}^\otimes \rightarrow \mathcal{O}^\otimes$ is a map of ∞ -operads. $\text{Alg}_{\mathcal{P}^\otimes/\mathcal{O}^\otimes}(\mathcal{M}^\otimes)$: ∞ -category of \mathcal{P} -algebra objects.
- $\text{CAlg}(\mathcal{M}^\otimes)$: ∞ -category of commutative algebra objects in a symmetric monoidal ∞ -category $\mathcal{M}^\otimes \rightarrow \text{N}(\text{Fin}_*)$. When the symmetric monoidal structure is clear, we usually write $\text{CAlg}(\mathcal{M})$ for $\text{CAlg}(\mathcal{M}^\otimes)$.
- CAlg_R : ∞ -category of commutative algebra objects in the symmetric monoidal ∞ -category Mod_R^\otimes where R is a commutative ring spectrum. When R is the sphere spectrum \mathbb{S} , we set $\text{CAlg} = \text{CAlg}_{\mathbb{S}}$. When R is the Eilenberg-MacLane spectrum Hk with k a ring, then we write CAlg_k for CAlg_R . If k is a field of characteristic zero, the ∞ -category CAlg_k is equivalent to the ∞ -category obtained from the model category of commutative differential graded k -algebras by inverting quasi-isomorphisms (cf. [31, 7.1.4.11]). Therefore we often refer to objects in CAlg_k as commutative differential graded algebras.
- $\text{Mod}_A^\otimes(\mathcal{M}^\otimes) \rightarrow \text{N}(\text{Fin}_*)$: symmetric monoidal ∞ -category of A -module objects, where \mathcal{M}^\otimes is a symmetric monoidal ∞ -category such that (1) the underlying ∞ -category admits a colimit for any simplicial diagram, and (2) its tensor product functor $\mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ preserves colimits of simplicial diagrams separately in each variable. Here A belongs to $\text{CAlg}(\mathcal{M}^\otimes)$. cf. [31, 3.3.3, 4.5.2].

Definition 1.8. Let \mathcal{C} be a stable presentable ∞ -category. Let $\{C_\alpha\}_{\alpha \in A}$ be a small set of objects in \mathcal{C} . We say that $\{C_\alpha\}_{\alpha \in A}$ generates \mathcal{C} as a stable presentable ∞ -category if \mathcal{C} is the smallest stable subcategory which contains $\{C_\alpha\}_{\alpha \in A}$ and is closed under small coproducts.

Suppose that \mathcal{C}^\otimes is a symmetric monoidal stable presentable ∞ -category whose tensor product $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ preserves small colimits separately in each variable. We say that $\{C_\alpha\}_{\alpha \in A}$ generates \mathcal{C}^\otimes as a symmetric monoidal stable presentable ∞ -category if \mathcal{C} is the smallest stable subcategory which contains the unit object and $\{C_\alpha\}_{\alpha \in A}$ and is closed under small coproducts and tensor product. (We remark that any stable ∞ -category which has small coproducts admits all small colimits.)

Remark 1.9. If each object C_α is compact and $\{C_\alpha\}_{\alpha \in A}$ generates \mathcal{C} as a stable presentable ∞ -category, we say that the stable presentable ∞ -category \mathcal{C} is compactly generated. This notion is compatible with the notion of compactly generated triangulated category. Namely, the compactness of C_α in \mathcal{C} and that in the triangulated category $\text{h}(\mathcal{C})$ coincide, and $\text{h}(\mathcal{C})$ is the smallest triangulated subcategory of $\text{h}(\mathcal{C})$ which contains $\{C_\alpha\}_{\alpha \in A}$ and is closed under small coproducts if and only if $\{C_\alpha\}_{\alpha \in A}$ generates \mathcal{C} as a stable presentable ∞ -category. In addition, if each object C_α is compact, these conditions are equivalent to the following: for any $C \in \mathcal{C}$, the vanishing $\text{Hom}_{\text{h}(\mathcal{C})}(C_\alpha, C[r]) = 0$ for any pair $(\alpha, r) \in A \times \mathbb{Z}$ implies $C \simeq 0$. Our reference are [37, 2.2.1], [31, 1.4.4.3].

2. PRELIMINARIES ON STACKS AND QUASI-COHERENT COMPLEXES

In this Section, we will recall some definitions and prepare several results concerning derived stacks, symmetric monoidal stable ∞ -categories, etc.

2.1. Derived stacks. Let k be a field. Let CAlg_k be the ∞ -category of commutative ring spectra over the Eilenberg-MacLane spectrum Hk . Set $\mathrm{Aff}_k := \mathrm{CAlg}_k^{\mathrm{op}}$. We refer to Aff_k as the ∞ -category of affine derived schemes over k . We denote by $\mathrm{Spec} R$ the object in Aff_k corresponding to R in CAlg_k . We say that a functor $F : \mathrm{Aff}_k^{\mathrm{op}} \rightarrow \widehat{\mathcal{S}}$ is an étale sheaf if

- for any finite coproduct $\sqcup_{i \in I} \mathrm{Spec} R_i$ in Aff_k , $F(\sqcup_{i \in I} \mathrm{Spec} R_i) \simeq \prod_{i \in I} F(\mathrm{Spec} R_i)$,
- for any étale hypercovering $\mathrm{Spec} B^\bullet \rightarrow \mathrm{Spec} A$, $F(A) \simeq \varprojlim_n F(B^n)$.

Here an étale hypercovering of $\mathrm{Spec} A$ is an augmented simplicial diagram of derived affine schemes $\mathrm{Spec} B^\bullet \rightarrow \mathrm{Spec} A$ such that for any $n \geq 0$, $\mathrm{Spec} B^n \rightarrow (\mathrm{cosk}_{n-1} \mathrm{Spec} B^\bullet)_n$ is étale surjective and $\mathrm{Spec} B^0 \rightarrow \mathrm{Spec} A$ is étale surjective. Let $\mathrm{Sh}(\mathrm{Aff}_k)$ be the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_k, \widehat{\mathcal{S}})$ spanned by étale sheaves. By Yoneda Lemma, there is a fully faithful functor $\mathrm{Aff}_k \rightarrow \mathrm{Fun}(\mathrm{CAlg}_k, \widehat{\mathcal{S}})$. The essential image is contained in $\mathrm{Sh}(\mathrm{Aff}_k)$.

A sheaf $X : \mathrm{CAlg}_k \rightarrow \widehat{\mathcal{S}}$ is a derived stack if there is a groupoid object $X_\bullet : \mathrm{N}(\Delta)^{\mathrm{op}} \rightarrow \mathrm{Aff}_k$ (see e.g. [41, 1.3.1.6], [30, 6.1.2.7] for groupoid objects) such that X is equivalent to a colimit of the composite $\mathrm{N}(\Delta)^{\mathrm{op}} \rightarrow \mathrm{Aff}_k \rightarrow \mathrm{Sh}(\mathrm{Aff}_k)$. We refer to X_\bullet as a presentation of X . A morphism $X \rightarrow Y$ of derived stacks is a morphism in $\mathrm{Sh}(\mathrm{Aff}_k)$. A morphism $X \rightarrow Y$ in $\mathrm{Sh}(\mathrm{Aff}_k)$ is said to be affine if for any $\mathrm{Spec} R \rightarrow Y$, the fiber product $\mathrm{Spec} R \times_Y X$ belongs to Aff_k . If X is a sheaf, Y is a derived stack and $f : X \rightarrow Y$ is affine, then X is a derived stack. The class of derived stacks is closed under products. A derived stack has affine diagonal. Our definition of derived stacks coincides with that of [25] and fits in nicely with our purpose, and it follows along the line similar to Toën-Vezzosi [41, 1.3.4]. But it is slightly different from the definitions given in [32], [41].

Fix convention of algebraic groups and their representations: By an algebraic group, we mean an affine group scheme of finite type over a field k . An affine group scheme over a field k is a pro-algebraic group over k . A representation of an affine group scheme $G = \mathrm{Spec} B$ over k is an (left or right) action of G on a k -vector space V , that is determined by the rule assigning to each k -algebra R and $g \in G(R)$ an isomorphism $\phi_g : V \otimes_k R \xrightarrow{\sim} V \otimes_k R$ of R -modules in the functorial fashion. Equivalently, a representation is a coaction $V \rightarrow V \otimes_k B$ of the commutative Hopf algebra B on V . As is well-known, every representation is a filtered colimit of finite-dimensional representations.

Let G be a usual affine group scheme over k . Then it gives rise to a group object $D_G : \mathrm{N}(\Delta)^{\mathrm{op}} \rightarrow \mathrm{Aff}_k$ given by $[n] \mapsto G^{\times n}$. We denote by BG the colimit of this group object in $\mathrm{Sh}(\mathrm{Aff}_k)$ and refer to BG as the classifying stack of G .

A derived stack X is said to be a quotient stack by action of G if there exist a presentation $X_\bullet : \mathrm{N}(\Delta)^{\mathrm{op}} \rightarrow \mathrm{Aff}_k$ of X and a natural transformation $X \rightarrow D_G$ such that for any $[m] \rightarrow [n]$, the diagram

$$\begin{array}{ccc} X_\bullet([n]) & \longrightarrow & X_\bullet([m]) \\ \downarrow & & \downarrow \\ D_G([n]) & \longrightarrow & D_G([m]) \end{array}$$

is the pullback square. Put $\mathrm{Spec} A = X_\bullet([0])$. In this case, we often write $[\mathrm{Spec} A/G]$ for the quotient stack.

2.2. Symmetric monoidal structure. We first recall briefly the notion of symmetric monoidal ∞ -categories. Let $\xi_{n,i} : \langle n \rangle \rightarrow \langle 1 \rangle$ be the map in Fin_* such that $\xi_{n,i}(j)$ is 1 if $j = i$ and $*$ if $j \neq i$. A symmetric monoidal ∞ -category is defined to be a coCartesian fibration $p : \mathcal{C}^\otimes \rightarrow \mathrm{N}(\mathrm{Fin}_*)$ such that

$$(\xi_{n,1})_* \times \dots \times (\xi_{n,n})_* : \mathcal{C}_n \rightarrow \mathcal{C}_1 \times \dots \times \mathcal{C}_1$$

is an equivalence for each $n \geq 0$. Here $\mathcal{C}_n := p^{-1}(\langle n \rangle)$. By convention, $\mathcal{C}_0 \simeq \Delta^0$. We refer to \mathcal{C}_1 as the underlying ∞ -category (but we usually denote by \mathcal{C} the underlying ∞ -category).

For ease of notation, we usually write \mathcal{C}^\otimes for $\mathcal{C}^\otimes \rightarrow \mathbf{N}(\mathbf{Fin}_*)$. For two symmetric monoidal ∞ -categories $p : \mathcal{C}^\otimes \rightarrow \mathbf{N}(\mathbf{Fin}_*)$ and $q : \mathcal{D}^\otimes \rightarrow \mathbf{N}(\mathbf{Fin}_*)$, a symmetric monoidal functor $\mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ is a map of coCartesian fibrations $\mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ over $\mathbf{N}(\mathbf{Fin}_*)$ which carries p -coCartesian edges to q -coCartesian edges.

We say that an object C in a symmetric monoidal ∞ -category \mathcal{C}^\otimes is dualizable if there exist an object C^\vee and two morphisms $e : C \otimes C^\vee \rightarrow 1$ and $c : 1 \rightarrow C \otimes C^\vee$ with 1 a unit such that the composition

$$C \xrightarrow{\text{id}_C \otimes c} C \otimes C^\vee \otimes C \xrightarrow{e \otimes \text{id}_C} C \quad \text{and} \quad C^\vee \xrightarrow{c \otimes \text{id}_{C^\vee}} C^\vee \otimes C \otimes C^\vee \xrightarrow{\text{id}_{C^\vee} \otimes e} C^\vee$$

are equivalent to the identity of C and the identity of C^\vee respectively. The symmetric monoidal structure of \mathcal{C} induces that of the homotopy category $\mathbf{h}(\mathcal{C})$. If we consider C to be an object also in $\mathbf{h}(\mathcal{C})$, then C is dualizable in \mathcal{C} if and only if C is dualizable in $\mathbf{h}(\mathcal{C})$.

Let $\text{Cat}_\infty^{\text{Sym}}$ denote the ∞ -category of symmetric monoidal small ∞ -categories, that is obtained from the simplicial category of symmetric monoidal ∞ -categories (regarded as coCartesian fibrations) whose morphisms are symmetric monoidal functors. Using the straightening functor [30, 3.2] and [30, 4.2.4.4] we have a fully faithful functor

$$\text{Cat}_\infty^{\text{Sym}} \rightarrow \text{Fun}(\mathbf{N}(\mathbf{Fin}_*), \text{Cat}_\infty).$$

The essential image is spanned by commutative monoid objects (i.e., E_∞ -monoid objects). If we equip Cat_∞ with the symmetric monoidal structure given by Cartesian product, then a commutative monoid object amounts to a commutative algebra object. Thus we have a natural categorical equivalence $\text{Cat}_\infty^{\text{Sym}} \simeq \text{CAlg}(\text{Cat}_\infty)$. We often think of a symmetric monoidal small ∞ -category as an object in $\text{CAlg}(\text{Cat}_\infty)$.

Let Pr^{L} be the subcategory of $\widehat{\text{Cat}}_\infty$ which consists of presentable ∞ -categories and whose edges (i.e. morphisms) are colimit-preserving functors. The ∞ -category Pr^{L} inherits a symmetric monoidal structure (see [31, 4.8.1.14, 4.8.1.16]). For two presentable ∞ -categories \mathcal{C} and \mathcal{D} , the tensor product $\mathcal{C} \otimes \mathcal{D}$ is given by $\text{Fun}^{\text{R}}(\mathcal{C}^{\text{op}}, \mathcal{D})$, where $\text{Fun}^{\text{R}}(-, -)$ denotes the full subcategory of $\text{Fun}(-, -)$ spanned by limit-preserving functors. According to [31, 4.8.1.16] and the proof, the tensor product $\mathcal{C} \otimes \mathcal{D}$ satisfies the following universal property: it admits a functor $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C} \otimes \mathcal{D}$ such that the composition induces a fully faithful functor

$$\text{Map}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \rightarrow \text{Map}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

whose essential image is spanned by functors which preserve (small) colimits separately in each variable. The ∞ -category \mathcal{S} of (small) spaces is a unit object in Pr^{L} . By [31, 4.1.16] and [30, 5.5.3.18], the tensor product $\text{Pr}^{\text{L}} \times \text{Pr}^{\text{L}} \rightarrow \text{Pr}^{\text{L}}$ preserves colimits separately in each variable.

A symmetric monoidal presentable ∞ -category \mathcal{C}^\otimes whose tensor product preserves (small) colimits separately in each variable can be viewed as a commutative algebra object in the symmetric monoidal ∞ -category $(\text{Pr}^{\text{L}})^\otimes$ in the same way that a symmetric monoidal small ∞ -category can be viewed as an object in $\text{CAlg}(\text{Cat}_\infty)$. Namely, \mathcal{C}^\otimes belongs to $\text{CAlg}(\text{Pr}^{\text{L}})$. A morphism in $\text{CAlg}(\text{Pr}^{\text{L}})$ corresponds to a symmetric monoidal functor which preserves (small) colimits. Let R be a commutative ring spectrum and Mod_R^\otimes the symmetric monoidal (stable) ∞ -category of R -module spectra. Since Mod_R^\otimes lies in $\text{CAlg}(\text{Pr}^{\text{L}})$, we can consider the symmetric monoidal ∞ -category $\text{Mod}_{\text{Mod}_R^\otimes}^\otimes(\text{Pr}^{\text{L}})$ of Mod_R^\otimes -module objects. We write Pr_R^{L} for $\text{Mod}_{\text{Mod}_R^\otimes}^\otimes(\text{Pr}^{\text{L}})$. We shall refer to an object in Pr_R^{L} as an R -linear presentable ∞ -category and refer to Pr_R^{L} as the ∞ -category of R -linear presentable ∞ -categories. Similarly, we shall refer to an object in $\text{CAlg}(\text{Pr}_R^{\text{L}})$ as an R -linear symmetric monoidal presentable ∞ -category and refer to $\text{CAlg}(\text{Pr}_R^{\text{L}})$ as the ∞ -category of R -linear symmetric monoidal presentable ∞ -categories. A morphism in $\text{CAlg}(\text{Pr}_R^{\text{L}})$ will be referred to as an R -linear symmetric monoidal functor. Consider the case of R is the sphere spectrum \mathbb{S} . By [31, 4.8.2.18] the forgetful functor $\text{Pr}_\mathbb{S}^{\text{L}} \rightarrow \text{Pr}^{\text{L}}$ can be regarded as the fully faithful embedding of the full subcategory

spanned by stable presentable ∞ -categories (recall that \mathbb{S} denotes the sphere spectrum). In particular, any R -linear presentable ∞ -category is stable. Let Sp denote the stable presentable ∞ -category of spectra. We denote by \otimes the smash product. The left adjoint of $\mathrm{Pr}_{\mathbb{S}}^{\mathrm{L}} \rightarrow \mathrm{Pr}^{\mathrm{L}}$ is given by $\mathrm{Pr}^{\mathrm{L}} \rightarrow \mathrm{Pr}_{\mathbb{S}}^{\mathrm{L}}$ which carries \mathcal{C} to $\mathcal{C} \otimes \mathrm{Sp} \simeq \varprojlim \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{S}_*)$ where $\mathcal{S}_* = \mathcal{S}_{\Delta^0/}$ is the ∞ -category of pointed spaces and the limit of the sequence of the loop space functor $\Omega_* : \mathcal{S}_* \rightarrow \mathcal{S}_*$ is taken in Pr^{L} . If R is the Eilenberg-MacLane spectrum HK for some (ordinary) commutative ring K , then we write $\mathrm{Pr}_K^{\mathrm{L}}$ for $\mathrm{Pr}_{HK}^{\mathrm{L}}$. In that case, we use the term “ K -linear presentable ∞ -category” instead of “ HK -linear presentable ∞ -category”. Recall that the homotopy category $\mathrm{h}(\mathcal{C})$ of a stable ∞ -category \mathcal{C} is a triangulated category, and in particular an additive category, see [31]. When \mathcal{C} is a K -linear presentable ∞ -category, the additive category $\mathrm{h}(\mathcal{C})$ is K -linear; every hom set $\mathrm{Hom}_{\mathrm{h}(\mathcal{C})}(C, D)$ has the structure of a K -vector space, and the composition $\mathrm{Hom}_{\mathrm{h}(\mathcal{C})}(D, E) \times \mathrm{Hom}_{\mathrm{h}(\mathcal{C})}(C, D) \rightarrow \mathrm{Hom}_{\mathrm{h}(\mathcal{C})}(C, E)$ is K -bilinear. The functor $\mathrm{Mod}_K \times \mathcal{C} \rightarrow \mathcal{C}$ given by Mod_K^{\otimes} -module structure induces an action of $K = \mathrm{Hom}_{\mathrm{h}(\mathrm{Mod}_K)}(1_K, 1_K)$ on $\mathrm{Hom}_{\mathrm{h}(\mathcal{C})}(C, D)$, where 1_K is a unit in Mod_K , and C and D belong to \mathcal{C} . It gives rise to the structure of a K -vector space

$$K \times \mathrm{Hom}_{\mathrm{h}(\mathcal{C})}(C, D) = \mathrm{Hom}_{\mathrm{h}(\mathrm{Mod}_K)}(1_K, 1_K) \times \mathrm{Hom}_{\mathrm{h}(\mathcal{C})}(C, D) \rightarrow \mathrm{Hom}_{\mathrm{h}(\mathcal{C})}(C, D)$$

where the right map is determined by $\mathrm{Mod}_K \times \mathcal{C} \rightarrow \mathcal{C}$. We easily see that the composition is K -bilinear.

2.3. Quasi-coherent complexes. Let X be a derived stack over k . Let X_{\bullet} be a presentation of X . Put $\mathrm{Spec} R^n = X_{\bullet}([n])$. Then $\mathrm{QC}^{\otimes}(X)$ is defined to be the limit $\varprojlim \mathrm{Mod}_{R^n}^{\otimes}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. This definition does not depend on the choice of X_{\bullet} . The construction $\mathrm{Spec} A \mapsto \mathrm{Mod}_A^{\otimes}$ gives rise to a functor $\mathrm{CAlg}_{\mathbb{S}} \rightarrow \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty})$. By $\mathrm{CAlg}_k \simeq (\mathrm{CAlg}_{\mathbb{S}})_k$, it gives rise to $\mathrm{QC}^{\otimes} : \mathrm{Aff}_k^{op} = \mathrm{CAlg}_k \rightarrow \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty})_{\mathrm{Mod}_k^{\otimes}/}$. The right Kan extension induces $\mathrm{Fun}(\mathrm{CAlg}_k, \widehat{\mathcal{S}})^{op} \rightarrow \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty})_{\mathrm{Mod}_k^{\otimes}/}$ which carries limits to limits. By descent theory, it factors through the sheafification and we obtain

$$\mathrm{QC}^{\otimes} : \mathrm{Sh}(\mathrm{Aff}_k) \rightarrow \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty})_{\mathrm{Mod}_k^{\otimes}/}$$

which carries X to $\mathrm{Mod}_k^{\otimes} \rightarrow \mathrm{QC}^{\otimes}(X)$. Since a small limit in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$ commutes with that in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty})$ and $\mathrm{Mod}_A^{\otimes} \in \mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$ for $A \in \mathrm{CAlg}_k$, thus for any derived stack X , $\mathrm{QC}^{\otimes}(X)$ belongs to $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$. We write \mathcal{O}_X for a unit object of the symmetric monoidal ∞ -category $\mathrm{QC}^{\otimes}(X)$. For a morphism $f : X \rightarrow Y$ of derived stacks (that is, a morphism as objects in $\mathrm{Sh}(\mathrm{Aff}_k)$), QC^{\otimes} induces a morphism $f^* : \mathrm{QC}^{\otimes}(Y) \rightarrow \mathrm{QC}^{\otimes}(X)$ in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$ (note that any morphism $A \rightarrow B$ in CAlg_k induces the base change functor $\mathrm{Mod}_A^{\otimes} \rightarrow \mathrm{Mod}_B^{\otimes}$ that belongs to $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$). By adjoint functor theorem there is a right adjoint $f_* : \mathrm{QC}(X) \rightarrow \mathrm{QC}(Y)$ of f^* . We shall refer to f^* and f_* as the pullback functor and the pushforward functor respectively. By [2, Proposition 3.10, Corollary 3.23], if X and Y are perfect (we recall this notion below), then $f_* : \mathrm{QC}(X) \rightarrow \mathrm{QC}(Y)$ preserves small colimits.

Perfect stacks. Let X be a derived stack over the base field k . We say that X is *perfect* if

- $\mathrm{QC}(X)$ is compactly generated,
- compact and dualizable objects in $\mathrm{QC}(X)$ coincide.

The notion of perfect stacks is introduced in [2, Definition 3.2, Proposition 3.9], and fundamental properties are extensively studied (see [2]). It turns out that the class of perfect derived stacks fits our purpose of use. For example, affine derived schemes $\mathrm{Spec} A$, the classifying stack BG , the quotient $[\mathrm{Spec} A/G]$ by a pro-reductive group scheme G , the product $X \times_k Y$ of perfect derived stacks are perfect derived stacks (cf. Example 2.4, [2, Proposition 3.21, 3.24]).

From model categories to ∞ -categories. We here recall a version of Dwyer-Kan localization in the context of ∞ -categories by which we can obtain ∞ -categories from model categories (see

[31, 1.3.4, 4.1.3], [20]). Let \mathbb{M} be a combinatorial model category (cf. [30]) and \mathbb{M}^c the full subcategory which consists of cofibrant objects. Then there is an ∞ -category $N_W(\mathbb{M}^c)$ and a functor $\xi : N(\mathbb{M}^c) \rightarrow N_W(\mathbb{M}^c)$ such that for any ∞ -category \mathcal{C} the composition induces a fully faithful functor

$$\mathrm{Map}(N_W(\mathbb{M}^c), \mathcal{C}) \rightarrow \mathrm{Map}(N(\mathbb{M}^c), \mathcal{C})$$

whose essential image consists of those functors $F : N(\mathbb{M}^c) \rightarrow \mathcal{C}$ such that F carry weak equivalences in $N(\mathbb{M}^c)$ to equivalences in \mathcal{C} . By Yoneda lemma, $N(\mathbb{M}^c) \rightarrow N_W(\mathbb{M}^c)$ is unique up to contractible space of choice. We shall refer to $N_W(\mathbb{M}^c)$ as the ∞ -category obtained from \mathbb{M} (or \mathbb{M}^c) by inverting weak equivalences. An explicit construction of $N_W(\mathbb{M}^c)$ is given by the hammock localization. More precisely, one model of $N_W(\mathbb{M}^c)$ is the simplicial nerve of (a fibrant replacement of) the hammock localization of \mathbb{M}^c . The homotopy category of $N_W(\mathbb{M}^c)$ coincides with the homotopy category of the model category \mathbb{M} . The ∞ -category $N_W(\mathbb{M}^c)$ is presentable. If \mathbb{M} is a stable model category, $N_W(\mathbb{M}^c)$ is stable (cf. [24]). If further \mathbb{M} is a symmetric monoidal model category, there is a symmetric monoidal ∞ -category $N_W^\otimes(\mathbb{M}^c)$ which belongs to $\mathrm{CAlg}(\mathrm{Pr}^L)$, and a symmetric monoidal colimit-preserving functor $\tilde{\xi} : N^\otimes(\mathbb{M}^c) \rightarrow N_W^\otimes(\mathbb{M}^c)$ which has ξ as the underlying functor and satisfies a similar universal property: for any symmetric monoidal ∞ -category \mathcal{C}^\otimes the composition induces a fully faithful functor $\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)}(N_W^\otimes(\mathbb{M}^c), \mathcal{C}^\otimes) \rightarrow \mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)}(N^\otimes(\mathbb{M}^c), \mathcal{C}^\otimes)$ whose essential image consists of those $F : N^\otimes(\mathbb{M}^c) \rightarrow \mathcal{C}^\otimes$ such that F carry weak equivalences in $N(\mathbb{M}^c)$ to equivalences in \mathcal{C} .

Let us consider the model category of chain complexes of representations. Let G be a pro-reductive group over a field k of characteristic zero. Let $\mathrm{Vect}(G)$ be the (symmetric monoidal) Grothendieck abelian category of (not necessarily finite dimensional) representations of G , that is, k -vector spaces equipped with actions of G . Let $\mathrm{Comp}(\mathrm{Vect}(G))$ be the symmetric monoidal category of (possibly unbounded) chain complexes of objects in $\mathrm{Vect}(G)$. Let \mathcal{G}_G be the set of finite coproducts of irreducible representations of G . Let $\mathcal{H} = \{0\}$. Then by the semi-simplicity of representations of G , the pair $(\mathcal{G}_G, \mathcal{H})$ is a flat descent structure in the sense of [6]. Consequently, there exists a combinatorial symmetric monoidal model structure on $\mathrm{Comp}(\mathrm{Vect}(G))$ such that (i) weak equivalences are exactly quasi-isomorphisms, and (ii) coproducts of objects in \mathcal{G} are cofibrants [6]. Let $\mathcal{D}^\otimes(BG)$ denote the symmetric monoidal presentable ∞ -category obtained from the full subcategory $\mathrm{Comp}(\mathrm{Vect}(G))^c$ of cofibrant objects by inverting weak equivalences. Since $\mathrm{Comp}(\mathrm{Vect}(G))$ admits a left adjoint symmetric monoidal functor $\mathrm{Comp}(k) \rightarrow \mathrm{Comp}(\mathrm{Vect}(G)); k \mapsto k$, inverting weak equivalence induces a symmetric monoidal colimit-preserving functor $\mathrm{Mod}_k^\otimes \rightarrow \mathcal{D}^\otimes(BG)$. Hence $\mathcal{D}^\otimes(BG)$ belongs to $\mathrm{CAlg}(\mathrm{Pr}_k^L) \simeq \mathrm{CAlg}(\mathrm{Pr}_k^L)_{\mathrm{Mod}_k^\otimes}$. There exists a natural equivalence $\mathcal{D}^\otimes(BG) \simeq \mathrm{QC}^\otimes(BG)$; see e.g. [26, Lemma 4.14] (in loc. cit., the reductive algebraic case is treated, but that applies mutatis-mutandis in the case of pro-reductive groups). We often write $\mathrm{Rep}^\otimes(G)$ for $\mathrm{QC}^\otimes(BG)$.

Relatively affine stacks. Let us review derived stacks that are affine over a base derived stack. Let $X : \mathrm{CAlg}_k \rightarrow \widehat{\mathcal{S}}$ be a functor and Aff_X the full subcategory of the overcategory $\mathrm{Fun}(\mathrm{CAlg}_k, \widehat{\mathcal{S}})_{/X}$ spanned by objects $U \rightarrow X$ affine over X , i.e., those objects $U \rightarrow X$ such that for any $\mathrm{Spec} A \rightarrow X$ the fiber product $\mathrm{Spec} A \times_X U$ lies in Aff_k . Let X be a derived stack over k . Let us observe that $\mathrm{Aff}_X \simeq \mathrm{CAlg}(\mathrm{QC}(X))^{op}$. Consider the functor $\mathrm{Fun}(\mathrm{CAlg}_k, \widehat{\mathcal{S}})^{op} \rightarrow \widehat{\mathrm{Cat}}_\infty$ given by $U \mapsto \mathrm{Fun}(\mathrm{CAlg}_k, \widehat{\mathcal{S}})_{/U}$ (see the construction before [30, 6.1.1.2] for the precise formulation). For $f : V \rightarrow U$, we let $f^* : \mathrm{Fun}(\mathrm{CAlg}_k, \widehat{\mathcal{S}})_{/U} \rightarrow \mathrm{Fun}(\mathrm{CAlg}_k, \widehat{\mathcal{S}})_{/V}$ be the functor induced by the base change $T \mapsto T \times_U V$. We here abuse notation by denoting by $\widehat{\mathrm{Cat}}_\infty$ the ∞ -category of super-large ∞ -categories. Since colimits in the ∞ -topos $\mathrm{Fun}(\mathrm{CAlg}_k, \widehat{\mathcal{S}})$ are universal [30, 6.1.3.9], for any colimit $\varinjlim_{\lambda \in I} V_\lambda \simeq U$ in $\mathrm{Fun}(\mathrm{CAlg}_k, \widehat{\mathcal{S}})$, we have $\mathrm{Fun}(\mathrm{CAlg}_k, \widehat{\mathcal{S}})_{/U} \simeq \varprojlim_{\lambda \in I} \mathrm{Fun}(\mathrm{CAlg}_k, \widehat{\mathcal{S}})_{/V_\lambda}$. Thus $\mathrm{Aff}_U \simeq \varprojlim_{\lambda \in I} \mathrm{Aff}_{V_\lambda}$. It follows that

the functor $\Phi : \text{Fun}(\text{CAlg}_k, \widehat{\mathcal{S}})^{op} \rightarrow \widehat{\text{Cat}}_\infty$ given by $X \mapsto \text{Aff}_X$ is limit-preserving. When $X = \text{Spec } R$, $\text{Aff}_X = \text{CAlg}_R^{op} = \text{CAlg}(\text{QC}(X))^{op}$. Consequently, Φ is a right Kan extension of the functor $\phi : \text{CAlg}_k \rightarrow \widehat{\text{Cat}}_\infty$ given by $R \mapsto \text{CAlg}_R^{op}$. Hence by descent theory $\Phi : \text{Fun}(\text{CAlg}_k, \widehat{\mathcal{S}})^{op} \rightarrow \widehat{\text{Cat}}_\infty$ factors through the sheafification $\text{Fun}(\text{CAlg}_k, \widehat{\mathcal{S}}) \rightarrow \text{Sh}(\text{Aff}_k)$. It gives rise to $\text{Sh}(\text{Aff}_k)^{op} \rightarrow \widehat{\text{Cat}}_\infty$.

Let $X_\bullet : \text{N}(\Delta)^{op} \rightarrow \text{Aff}_k$ be a presentation of X and put $\text{Spec } R^n = X_\bullet([n])$. It follows that

$$\begin{aligned} \text{Aff}_X &\simeq \varprojlim_{[n] \in \Delta} \text{Aff}_{X([n])} \\ &\simeq \varprojlim_{[n] \in \Delta} \text{CAlg}(\text{QC}(\text{Spec } R^n))^{op} \\ &\simeq \text{CAlg}(\varprojlim_{[n] \in \Delta} \text{QC}(\text{Spec } R^n))^{op} \\ &\simeq \text{CAlg}(\text{QC}(X))^{op}. \end{aligned}$$

Thus $\text{Aff}_X \simeq \text{CAlg}(\text{QC}(X))^{op}$.

Let G be an affine group scheme over k . Suppose that $X = BG$ and $D_G : \text{N}(\Delta)^{op} \rightarrow \text{Aff}_k$ is the corresponding group object. Let A be an object in $\text{CAlg}(\text{QC}(BG))$. Let us describe the corresponding object in Aff_{BG} as a quotient stack denoted by $[\text{Spec } A/G]$. To this end, use the equivalence $\varprojlim_{[n] \in \Delta} \text{Aff}_{D_G([n])} \xrightarrow{\sim} \text{Aff}_{BG}$. For ease of notation put $G_n := D_G([n]) \simeq G^{\times n}$. Let $\text{Aff}_{G_\bullet} \rightarrow \text{N}(\Delta)^{op}$ be the coCartesian fibration corresponding to $[n] \mapsto \text{Aff}_{G_n}$ via the unstraightening functor (cf. [30, 3.2]). The limit $\varprojlim_{[n] \in \Delta} \text{Aff}_{G_n}$ is defined as the full subcategory of $\text{Fun}_{\text{N}(\Delta)^{op}}(\text{N}(\Delta)^{op}, \text{Aff}_{G_\bullet})$ spanned by functors which send all edges to coCartesian edges. The homomorphism $G \rightarrow \text{Spec } k$ of group schemes to the trivial group scheme $\text{Spec } k$ gives rise to a maps of coCartesian fibrations

$$\begin{array}{ccc} \text{Aff}_k \times \text{N}(\Delta)^{op} & \xrightarrow{\quad} & \text{Aff}_{G_\bullet} \\ & \searrow \text{pr}_2 \quad \swarrow & \\ & \text{N}(\Delta)^{op} & \end{array}$$

such that each fiber $\text{Aff}_k \rightarrow \text{Aff}_{G_n}$ is given by $A \mapsto A \otimes R^n$. Here $G_n = \text{Spec } R^n$. This map carries coCartesian edges to coCartesian edges. By the relative adjoint functor theorem [31, 7.3.2.7] there is a right adjoint functor $c : \text{Aff}_{G_\bullet} \rightarrow \text{Aff}_k \times \text{N}(\Delta)^{op}$ relative to $\text{N}(\Delta)^{op}$. (On each fiber c induces $\text{Aff}_{G_n} \rightarrow \text{Aff}_k$ determined by the composition with $G_n \rightarrow \text{Spec } k$.) The composition with

$$\text{Aff}_{G_\bullet} \xrightarrow{c} \text{Aff}_k \times \text{N}(\Delta)^{op} \xrightarrow{\text{pr}_2} \text{Aff}_k \hookrightarrow \text{Sh}(\text{Aff}_k)$$

induces

$$\text{Fun}_{\text{N}(\Delta)^{op}}(\text{N}(\Delta)^{op}, \text{Aff}_{G_\bullet}) \xrightarrow{\eta} \text{Fun}(\text{N}(\Delta)^{op}, \text{Aff}_k) \rightarrow \text{Fun}(\text{N}(\Delta)^{op}, \text{Sh}(\text{Aff}_k)) \xrightarrow{\text{colim}} \text{Sh}(\text{Aff}_k).$$

Note that it carries the final object in $\varprojlim_{[n] \in \Delta} \text{Aff}_{G_n}$ to BG . We then obtain $\varprojlim_{[n] \in \Delta} \text{Aff}_{G_n} \rightarrow \text{Sh}(\text{Aff}_k)_{/BG}$ which induces the equivalence $\varprojlim_{[n] \in \Delta} \text{Aff}_{G_n} \simeq \text{Aff}_{BG}$. Unwinding the construction, we see that any $X \in \varprojlim_{[n] \in \Delta} \text{Aff}_{G_n}$ gives rise to $\eta(X) : \text{N}(\Delta)^{op} \rightarrow \text{Aff}_k$ and a natural transformation $\eta(X) \rightarrow D_G$. We easily see that $\eta(X) \rightarrow D_G$ satisfies the axiom of quotient stacks. If we write $A \in \text{CAlg}_k$ for the image of $A \in \text{CAlg}(\text{QC}(BG))$, then the colimit of $\text{N}(\Delta)^{op} \xrightarrow{\eta(X)} \text{Aff}_k \hookrightarrow \text{Sh}(\text{Aff}_k)$ is a quotient stack $[\text{Spec } A/G]$.

Return to the case of an arbitrary derived stack X with a presentation X_\bullet . Let $A \in \text{CAlg}(\text{QC}(X))$ and $p : W \rightarrow X \in \text{Aff}_X$ the corresponding object. We relate $\text{Mod}_A^\otimes(\text{QC}(X))$ with $\text{QC}^\otimes(W)$. Then we have the adjoint pair $p^* : \text{QC}(X) \rightleftarrows \text{QC}(W) : p_*$. Let \mathcal{O}_W be the unit of $\text{QC}(W)$ and set $A' = p_*(\mathcal{O}_W)$. We regard A' as an object in $\varprojlim_{[n] \in \Delta} \text{CAlg}(\text{QC}(\text{Spec } R^n))$,

then by base change formula [2, Proposition 3.10] we see that $A' = A$. Consider the symmetric monoidal functors

$$\mathrm{Mod}_A^\otimes(\mathrm{QC}(X)) \xrightarrow{p^*} \mathrm{Mod}_{p^*(A)}^\otimes(\mathrm{QC}(W)) \rightarrow \mathrm{QC}^\otimes(W)$$

where the right functor is induced by the base change by the counit map $p^*p_*(\mathcal{O}_W) \rightarrow \mathcal{O}_W$. We will prove in Proposition 2.3 that the above composite is an equivalence.

The following result is useful:

Proposition 2.1. *Let \mathcal{C}^\otimes and \mathcal{D}^\otimes be objects in $\mathrm{CAlg}(\mathrm{Pr}_\mathbb{S}^\mathrm{L})$, i.e., symmetric monoidal stable presentable ∞ -categories whose tensor product preserves small colimits separately in each variable. Let $F : \mathcal{D}^\otimes \rightarrow \mathcal{C}^\otimes$ be a symmetric monoidal functor which preserves small colimits. Let $G : \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ be a lax symmetric monoidal right adjoint functor (which exists by the relative adjoint functor theorem [31, 7.3.2.6]). Let $1_{\mathcal{C}}$ be a unit of \mathcal{C} (thus $1_{\mathcal{C}} \in \mathrm{CAlg}(\mathcal{C})$) and $B := G(1_{\mathcal{C}}) \in \mathrm{CAlg}(\mathcal{D})$. Consider the composite of symmetric monoidal colimit-preserving functors*

$$F' : \mathrm{Mod}_B^\otimes(\mathcal{D}) \xrightarrow{F} \mathrm{Mod}_{F(B)}^\otimes(\mathcal{C}) \rightarrow \mathrm{Mod}_{1_{\mathcal{C}}}^\otimes(\mathcal{C}) \simeq \mathcal{C}^\otimes$$

where the right functor is determined by the counit map $F \circ G(1_{\mathcal{C}}) \rightarrow 1_{\mathcal{C}}$. Suppose that

- (1) *there is a small set $\{I_\lambda\}_{\lambda \in \Lambda}$ of compact and dualizable objects of \mathcal{D} which generates \mathcal{D} as a stable presentable ∞ -category,*
- (2) *each $F(I_\lambda)$ is compact, and $\{F(I_\lambda)\}_{\lambda \in \Lambda}$ generates \mathcal{C} as a stable presentable ∞ -category.*

Then F' is an equivalence.

If F satisfies (1) and (2) in Proposition 2.1 we say that F is perfect. Let $G' : \mathcal{C}^\otimes \rightarrow \mathrm{Mod}_B^\otimes(\mathcal{D}^\otimes)$ be a lax symmetric monoidal functor which is a right adjoint functor of F' . The existence of the right adjoint functor follows from the relative version of adjoint functor theorem (see [31, 7.3.2.6]). Therefore we have a diagram

$$\begin{array}{ccc} \mathcal{D}^\otimes & \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} & \mathcal{C}^\otimes \\ \begin{array}{c} \uparrow U \\ \downarrow R \end{array} & \begin{array}{c} \nearrow F' \\ \searrow G' \end{array} & \\ \mathrm{Mod}_B^\otimes(\mathcal{D}^\otimes) & & \end{array}$$

where U is the forgetful functor and R assigns a free left B -module $B \otimes M$ to any $M \in \mathcal{D}^\otimes$. All functors are exact. The composite $F' \circ R : \mathcal{D}^\otimes \rightarrow \mathcal{C}^\otimes$ is equivalent to F as symmetric monoidal functors.

Lemma 2.2. *Suppose that $\{I_\lambda\}_{\lambda \in \Lambda}$ is a small set of compact objects which generates \mathcal{D} as a stable presentable ∞ -category. Then $\{R(I_\lambda)\}_{\lambda \in \Lambda}$ is a set of compact objects which generates $\mathrm{Mod}_B(\mathcal{D}^\otimes)$ as a stable presentable ∞ -category.*

Proof. We first show that $R(I_\lambda)$ is compact. Let $\varinjlim N_i$ is a filtered colimit in $\mathrm{Mod}_B(\mathcal{D}^\otimes)$. Note that by [31, 4.2.3.5] U preserves colimits and thus $\varinjlim U(N_i) \simeq U(\varinjlim N_i)$. Then we have natural equivalences

$$\begin{aligned} \mathrm{Map}_{\mathrm{Mod}_B(\mathcal{D}^\otimes)}(R(I_\lambda), \varinjlim N_i) &\simeq \mathrm{Map}_{\mathcal{D}}(I_\lambda, U(\varinjlim N_i)) \\ &\simeq \mathrm{Map}_{\mathcal{D}}(I_\lambda, \varinjlim U(N_i)) \\ &\simeq \varinjlim \mathrm{Map}_{\mathcal{D}}(I_\lambda, U(N_i)) \\ &\simeq \varinjlim \mathrm{Map}_{\mathrm{Mod}_B(\mathcal{D}^\otimes)}(R(I_\lambda), N_i) \end{aligned}$$

in \mathcal{S} . Notice that the third equivalence follows from the compactness of I_λ . By these equivalences, we conclude that $R(I_\lambda)$ is compact. It remains to prove that if $\mathrm{Ext}_{\mathrm{Mod}_B(\mathcal{D}^\otimes)}^n(R(I_\lambda), N) = 0$ for any $\lambda \in \Lambda$ and any integer $n \in \mathbb{Z}$, then N is zero. Since

$$\mathrm{Ext}_{\mathrm{Mod}_B(\mathcal{D}^\otimes)}^n(R(I_\lambda), N) \simeq \mathrm{Ext}_{\mathcal{D}}^n(I_\lambda, U(N)) \simeq \mathrm{Ext}_{\mathcal{D}}^n(I_\lambda, U(N)) = 0,$$

our claim follows from the fact that $\{I_\lambda\}_{\lambda \in \Lambda}$ is a compact generator and $\text{Mod}_B(\mathcal{D}^\otimes) \rightarrow \mathcal{D}$ is conservative. \square

Proof of Proposition 2.1. If F' is fully faithful, F' is also essentially surjective. In fact, if F' is fully faithful, the essential image of F' is the smallest stable subcategory of \mathcal{C} which has colimits and contains $F(I_\lambda)$ for all $\lambda \in \Lambda$. By the condition (2), the essential image of F' coincides with \mathcal{C} . Hence we will prove that F' is fully faithful. For this purpose, since F' is an exact functor between stable ∞ -categories $\text{Mod}_B(\mathcal{D}^\otimes)$ and \mathcal{C} , by [24, Lemma 5.8] it will suffice to show that F' induces a fully faithful functor between their homotopy categories. We will prove that F' induces a bijection

$$\alpha : \text{Hom}_{\text{Mod}_B(\mathcal{D}^\otimes)}(R(I_\lambda), R(\Sigma^n I_\mu)) \rightarrow \text{Hom}_{\mathcal{C}}(F'(R(I_\lambda)), F'(R(\Sigma^n I_\mu)))$$

where $\text{Hom}(-, -)$ indicates $\pi_0(\text{Map}(-, -))$ and n is an integer. Note that by adjunction, we have natural bijections

$$\begin{aligned} \text{Hom}_{\text{Mod}_B(\mathcal{D}^\otimes)}(R(I_\lambda), R(\Sigma^n I_\mu)) &\simeq \text{Hom}_{\mathcal{D}}(I_\lambda, U(R(\Sigma^n I_\mu))) \\ &\simeq \text{Hom}_{\mathcal{D}}(I_\lambda \otimes (\Sigma^n I_\mu)^\vee, G(1_{\mathcal{C}})). \end{aligned}$$

Here $(\Sigma^n I_\mu)^\vee$ is the dual of $\Sigma^n I_\mu$. On the other hand, we have natural bijections

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(F'(R(I_\lambda)), F'(R(\Sigma^n I_\mu))) &\simeq \text{Hom}_{\mathcal{C}}(F(I_\lambda), F(\Sigma^n I_\mu)) \\ &\simeq \text{Hom}_{\mathcal{C}}(F(I_\lambda \otimes (\Sigma^n I_\mu)^\vee), 1_{\mathcal{C}}). \end{aligned}$$

Also, by adjunction there is a bijection

$$\beta : \text{Hom}_{\mathcal{D}}(I_\lambda \otimes (\Sigma^n I_\mu)^\vee, G(1_{\mathcal{C}})) \rightarrow \text{Hom}_{\mathcal{C}}(F(I_\lambda \otimes (\Sigma^n I_\mu)^\vee), 1_{\mathcal{C}})$$

which carries $f : I_\lambda \otimes (\Sigma^n I_\mu)^\vee \rightarrow G(1_{\mathcal{C}})$ to $F(I_\lambda \otimes (\Sigma^n I_\mu)^\vee) \xrightarrow{F(f)} F(G(1_{\mathcal{C}})) \rightarrow 1_{\mathcal{C}}$ where the second morphism is the counit map. Therefore, it is enough to identify α with β through the natural bijections. Since F' is symmetric monoidal, by replacing I_λ and $\Sigma^n I_\mu$ by $I_\lambda \otimes (\Sigma^n I_\mu)^\vee$ and $1_{\mathcal{D}}$ respectively, we may and will assume that $\Sigma^n I_\mu = 1_{\mathcal{D}}$. According to the definition, α carries $f : R(I_\lambda) = G(1_{\mathcal{C}}) \otimes I_\lambda \rightarrow R(1_{\mathcal{D}}) = B$ to

$$1_{\mathcal{C}} \otimes_{F(G(1_{\mathcal{C}}))} F \circ U(f) : 1_{\mathcal{C}} \otimes_{F(G(1_{\mathcal{C}}))} F(G(1_{\mathcal{C}}) \otimes I_\lambda) \rightarrow 1_{\mathcal{C}} \otimes_{F(G(1_{\mathcal{C}}))} F(G(1_{\mathcal{C}}) \otimes 1_{\mathcal{C}}) \simeq 1_{\mathcal{C}}.$$

Unwinding the definitions, β sends $f : R(I_\lambda) \rightarrow R(1_{\mathcal{D}})$ to the composite

$$F(I_\lambda) \rightarrow F \circ U \circ R(I_\lambda) \rightarrow F \circ U \circ R(1_{\mathcal{D}}) = F(G(1_{\mathcal{C}})) \rightarrow 1_{\mathcal{C}} = F(1_{\mathcal{D}})$$

where the first functor is induced by the unit $\text{id} \rightarrow U \circ R$, the second functor is $F \circ U(f)$, and the third functor is induced by the counit $F \circ G \rightarrow \text{id}$. Note that $F(I_\lambda) \rightarrow F \circ U \circ R(I_\lambda)$ can be identified with $1_{\mathcal{C}} \otimes_{F(G(1_{\mathcal{C}}))} F(G(1_{\mathcal{C}}) \otimes I_\lambda) \rightarrow F(G(1_{\mathcal{C}}) \otimes I_\lambda)$ induced by the unit $1_{\mathcal{C}} \rightarrow F(G(1_{\mathcal{C}}))$ of $F(G(1_{\mathcal{C}})) \in \text{CAlg}(\mathcal{C}^\otimes)$. Now the desired identification with β follows from the fact that $F(I_\lambda) \rightarrow F \circ U \circ R(I_\lambda) \rightarrow 1_{\mathcal{C}} \otimes_{F(G(1_{\mathcal{C}}))} F(G(1_{\mathcal{C}}) \otimes I_\lambda) \simeq F(I_\lambda)$ is the identity (note that $1_{\mathcal{C}} \rightarrow F(G(1_{\mathcal{C}})) \rightarrow 1_{\mathcal{C}}$ is the identity).

Next we then apply the bijection α to conclude that F' is fully faithful. Since F' preserves colimits (in particular, exact), we see that if $N, M \in \text{Mod}_B(\mathcal{D}^\otimes)$ belongs to the smallest stable subcategory \mathcal{E} which contains $\{R(I_\lambda)\}_{\lambda \in \Lambda}$, then F' induces a bijection

$$\text{Hom}_{\text{Mod}_B(\mathcal{D}^\otimes)}(N, M) \rightarrow \text{Hom}_{\mathcal{C}}(F'(N), F'(M)).$$

There is a categorical equivalence $\text{Ind}(\mathcal{E}) \simeq \text{Mod}_B(\mathcal{D}^\otimes)$ which follows from Lemma 2.2 and [30, 5.3.5.11]. Again by [30, 5.3.5.11] and the fact that $F'(E)$ is compact for any $E \in \mathcal{E}$ (by condition (2)), a left Kan extension $\text{Ind}(\mathcal{E}) \rightarrow \mathcal{C}$ induced by $F' : \mathcal{E} \rightarrow \mathcal{C}$ (cf. [30, 5.3.5.10], [31, 4.8.1.13]) is fully faithful. This implies that F' is fully faithful. \square

Proposition 2.3. *We adopt the notation in the discussion before Proposition 2.1. Suppose that X is a perfect derived stack and $p : W \rightarrow X$ belongs to Aff_X . Put $A = p_*(\mathcal{O}_W)$. Then the composite*

$$\text{Mod}_A^\otimes(\text{QC}(X)) \rightarrow \text{QC}^\otimes(W)$$

given by $M \mapsto \mathcal{O}_W \otimes_{p^(A)} p^*(M)$ is an equivalence.*

Proof. By Proposition 2.1, it will suffice to prove that $p^* : \text{QC}(X) \rightarrow \text{QC}(W)$ is perfect; there is a set $\{I_\lambda\}_{\lambda \in \Lambda}$ of compact and dualizable objects in $\text{QC}(X)$ such that (i) $\text{QC}(X)$ is generated by $\{I_\lambda\}_{\lambda \in \Lambda}$ as a stable presentable ∞ -category, and (ii) $p^*(I_\lambda)$ is compact for any $\lambda \in \Lambda$ and $\text{QC}(W)$ is generated by $\{p^*(I_\lambda)\}_{\lambda \in \Lambda}$ as a stable presentable ∞ -category. Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be the set of compact objects in $\text{QC}(X)$. Then (i) is satisfied (X is perfect and I_λ are dualizable). Observe that $p^*(I_\lambda)$ is compact because $\text{Hom}_{\text{h}(\text{QC}(W))}(p^*(I_\lambda), \bigoplus_{j \in J} C_j) \simeq \text{Hom}_{\text{h}(\text{QC}(X))}(I_\lambda, p_*(\bigoplus_{j \in J} C_j)) \simeq \text{Hom}_{\text{h}(\text{QC}(X))}(I_\lambda, \bigoplus_{j \in J} p_*(C_j)) \simeq \bigoplus_{j \in J} \text{Hom}_{\text{h}(\text{QC}(X))}(I_\lambda, p_*(C_j))$. Let $X = \varinjlim_{[n]} \text{Spec } R^n$ be a presentation of X . Put $\text{Spec } A^0 := \text{Spec } R^0 \times_X W$. Note that p_* is conservative, i.e. $p_*(M) \simeq 0$ implies $M = 0$, since the pushforward $\text{Mod}_{A^0} = \text{QC}(\text{Spec } A^0) \rightarrow \text{QC}(\text{Spec } R^0) = \text{Mod}_{R^0}$ is conservative. Using the adjoint pair (p^*, p_*) and the conservativity we see that the vanishing $\text{Hom}_{\text{h}(\text{QC}(W))}(p^*(I_\lambda), C[r]) = 0$ for any $(\lambda, r) \in \Lambda \times \mathbb{Z}$ implies $C \simeq 0$. Therefore (ii) is satisfied. \square

Example 2.4. Suppose that G is a pro-reductive group over k . Then the set of (finite-dimensional) irreducible representations of G generates $\text{QC}(BG) \simeq \mathcal{D}(BG)$ as a stable presentable ∞ -category, and each irreducible representations is compact in $\text{QC}(BG)$. Hence $\text{QC}(BG)$ is compactly generated. Moreover, a unit object is compact, and it follows that every dualizable object is compact. Thus BG is a perfect derived stack. For $A \in \text{CAlg}(\text{QC}(BG))$, we have an natural equivalence $\text{Mod}_A^\otimes(\text{QC}(BG)) \simeq \text{QC}^\otimes([\text{Spec } A/G])$.

3. A UNIVERSAL CHARACTERIZATION OF REPRESENTATIONS OF GENERAL LINEAR GROUPS

Throughout this Section, k is a field of characteristic zero. Let \mathcal{C}^\otimes be a k -linear symmetric monoidal presentable ∞ -category. Let $\mathcal{C}_{\wedge, d}$ denote the full subcategory of d -dimensional wedge-finite objects in \mathcal{C} , and let $\mathcal{C}_{\wedge, d}^\simeq$ be the largest Kan subcomplex. Let $\text{Rep}^\otimes(\text{GL}_d) = \text{QC}^\otimes(B\text{GL}_d)$. The main purpose of this Section is to prove the following:

Theorem 3.1. *Let \mathcal{C}^\otimes be a k -linear symmetric monoidal (stable) presentable ∞ -category, i.e., an object in Pr_k^{L} . Then there exists a natural homotopy equivalence of spaces*

$$\text{Map}_{\text{CAlg}(\text{Pr}_k^{\text{L}})}(\text{Rep}^\otimes(\text{GL}_d), \mathcal{C}^\otimes) \rightarrow \mathcal{C}_{\wedge, d}^\simeq$$

which carries $f : \text{Rep}^\otimes(\text{GL}_d) \rightarrow \mathcal{C}^\otimes$ to the image $f(K)$ of the standard representation K of GL_d . That is, an object $C \in \mathcal{C}_{\wedge, d}$ corresponds to a k -linear symmetric monoidal functor $\text{Rep}^\otimes(\text{GL}_d) \rightarrow \mathcal{C}^\otimes$ that sends K to C .

Remark 3.2. By Theorem 3.1 every wedge-finite object is the image of the standard representation of GL_d for some $d \geq 0$ under a symmetric monoidal functor. The standard representation K is dualizable in $\text{Rep}^\otimes(\text{GL}_d)$, and any symmetric monoidal functor preserves dualizable objects. Hence every wedge-finite object is dualizable.

Remark 3.3. We use the assumption that the field k is characteristic zero in an essential way.

We define the category $B\Sigma$ as follows: Objects of $B\Sigma$ are finite sets, that is, $\bar{0}, \bar{1}, \dots, \bar{n} = \{1, \dots, n\}, \dots$. By convention $\bar{0}$ is the empty set. A morphism in $B\Sigma$ is a bijective map $\bar{n} \rightarrow \bar{m}$. Namely, $\text{Hom}_{B\Sigma}(\bar{n}, \bar{m})$ is isomorphic to the symmetric group Σ_n for $n \geq 0$, where Σ_0 is the group consisting of one element. If $n \neq m$, $\text{Hom}_{B\Sigma}(\bar{n}, \bar{m})$ is the empty set. Thus $B\Sigma$ is the coproduct $\sqcup_{n \geq 0} B\Sigma_n$ (in Cat_∞) where $B\Sigma_n$ is the category consisting of one object \bar{n} (regarded as a formal symbol) such that $\text{Hom}_{B\Sigma_n}(\bar{n}, \bar{n}) = \Sigma_n$. Let Vect_k be the category of k -vector spaces. We here

denote by $\text{Fun}(B\Sigma^{op}, \text{Vect}_k)$ the functor category. It is a Grothendieck abelian category; it is presentable (cf. [30, 5.5.3.6]) and monomorphisms are closed under filtered colimits.

Given an abelian category \mathcal{A} , we write $\text{Comp}(\mathcal{A})$ for the category of chain complexes of objects in \mathcal{A} . The category $\text{Comp}(\text{Fun}(B\Sigma^{op}, \text{Vect}_k))$ is isomorphic to the functor category $\text{Fun}(B\Sigma^{op}, \text{Comp}(k))$. Here for ease of notation we write $\text{Comp}(k)$ for $\text{Comp}(\text{Vect}_k)$. An object $E : B\Sigma^{op} \rightarrow \text{Comp}(k)$ corresponds to a symmetric sequence in the sense of [22, Section 6], that is,

$$(E_0, E_1, \dots, E_n, \dots)$$

where each chain complex $E_n = E(\bar{n})$ is endowed with right Σ_n -action. Recall from [22] the symmetric monoidal structure on $\text{Fun}(B\Sigma^{op}, \text{Comp}(k))$, the tensor product $E \otimes F$ for $E, F \in \text{Fun}(B\Sigma^{op}, \text{Comp}(k))$ is given by

$$\bar{l} \mapsto \coprod_{A \sqcup B = \bar{l}, A \cap B = \emptyset} E(A) \otimes F(B)$$

which is Σ_l -equivariantly isomorphic to

$$\bar{l} \mapsto \coprod_{n+m=l} E(\bar{n}) \otimes F(\bar{m}) \otimes_{k[\Sigma_n] \otimes_k k[\Sigma_m]} k[\Sigma_l]$$

on which Σ_l acts by the right multiplication. Here for a finite group G , $k[G]$ denotes the group algebra, and $E(\bar{n}) \otimes F(\bar{m})$ is considered to be a right $k[\Sigma_n] \otimes_k k[\Sigma_m]$ -module, and $k[\Sigma_l]$ is considered as a left $k[\Sigma_n \times \Sigma_m]$ -module through the natural inclusion $\Sigma_n \times \Sigma_m \subset \Sigma_l$. For any $a \geq 0$, we define a symmetric sequence $I^a = (I_n^a)_{n \geq 0}$ by $I_a^a = k[\Sigma_a]$ equipped with the right multiplication of Σ_a , and $I_n^a = 0$ for $n \neq a$. Then for any $a, b \geq 0$, the tensor product $I^a \otimes I^b$ is I^{a+b} , and the commutative constraint on $I_{a+b}^{a+b} = k[\Sigma_{a+b}]$ is defined by the left action of the permutation $(1, \dots, a, a+1, \dots, a+b) \mapsto (a+1, \dots, a+b, 1, \dots, a)$.

By using the machinery in [6], we equip $\text{Fun}(B\Sigma^{op}, \text{Comp}(k))$ with a combinatorial symmetric monoidal model structure. The class of weak equivalences are (exactly) termwise quasi-isomorphisms. Let \mathcal{G} be the set of finite coproducts of objects in $\text{Fun}(B\Sigma^{op}, \text{Vect}_k)$ which have the form $(E_n)_{n \geq 0}$ such that there is a non-negative integer i such that E_i is an irreducible k -linear Σ_n -representation, and $E_n = 0$ if $n \neq i$. Set $\mathcal{H} = 0$. Then by the representation theory of symmetric groups in characteristic zero and its semi-simplicity, we see that the pair $(\mathcal{G}, \mathcal{H})$ is a flat descent structure in the sense of [6]. According to [6, Theorem 2.5, Proposition 3.2], there is a proper combinatorial symmetric monoidal model structure on $\text{Fun}(B\Sigma^{op}, \text{Comp}(k))$ in which weak equivalences are termwise quasi-isomorphisms (we do not recall the cofibrations and fibrations, see [6]).

Let $\mathcal{D}^\otimes(B\Sigma, k) := \text{N}_W(\text{Fun}(B\Sigma^{op}, \text{Comp}(k))^c)$ be the symmetric monoidal presentable stable ∞ -category obtained from the full subcategory $\text{Fun}(B\Sigma^{op}, \text{Comp}(k))^c$ of cofibrant objects by inverting weak equivalences. The ∞ -category $\mathcal{D}(B\Sigma, k)$ is equivalent to $\text{N}_W(\text{Fun}(B\Sigma^{op}, \text{Comp}(k)))$. By [31, 1.3.4.25] and $B\Sigma \simeq \sqcup_{n \geq 0} B\Sigma_n$, there exist equivalences of ∞ -categories

$$\mathcal{D}(B\Sigma, k) \simeq \text{Fun}(B\Sigma^{op}, \text{Mod}_k) \simeq \prod_{n \geq 0} \text{Fun}(B\Sigma_n^{op}, \text{Mod}_k),$$

where $\text{Fun}(-, -)$ in the right and middle sides denotes the function complex. Here we abuse notation by indicating with $B\Sigma$ the nerves of $B\Sigma$.

Let us consider the functor category $\text{Fun}(B\Sigma_n^{op}, \text{Comp}(k))$, which we often identified with the category of chain complexes of k -linear representations, that is, k -vector spaces endowed with right actions of Σ_n . As in the case of $\text{Fun}(B\Sigma^{op}, \text{Comp}(k))$, $\text{Fun}(B\Sigma_n^{op}, \text{Comp}(k))$ admits a combinatorial symmetric monoidal model structure in which weak equivalences are exactly quasi-isomorphisms. Let $\mathcal{D}^\otimes(B\Sigma_n, k)$ be the symmetric monoidal presentable ∞ -category obtained from $\text{Fun}(B\Sigma_n^{op}, \text{Comp}(k))^c$ by inverting weak equivalences. The underlying ∞ -category $\mathcal{D}(B\Sigma_n, k)$ can be identified with $\text{Fun}(B\Sigma_n^{op}, \text{Mod}_k)$. The homotopy category of $\mathcal{D}(B\Sigma_n, k)$ is the (unbounded) derived category of k -linear representations of Σ_n . Since $\mathcal{D}(B\Sigma, k)$ is the

product of $\{\mathcal{D}(B\Sigma_n, k)\}_{n \geq 0}$, we often write $(E_n)_{n \geq 0}$ with $E_n \in \mathcal{D}(B\Sigma_n, k)$ for an object in $\mathcal{D}(B\Sigma, k)$. If we regard $E_n \in \mathcal{D}(B\Sigma_n, k)$ as an object in $\mathcal{D}(B\Sigma, k)$ in the obvious way, the coproduct $\bigoplus_{n \geq 0} E_n$ in $\mathcal{D}(B\Sigma, k)$ is $(E_n)_{n \geq 0}$ since

$$\mathrm{Map}(\bigoplus_{n \geq 0} E_n, (F_n)_{n \geq 0}) \simeq \prod_{n \geq 0} \mathrm{Map}(E_n, (F_n)_{n \geq 0}) \simeq \prod_{n \geq 0} \mathrm{Map}(E_n, F_n) = \mathrm{Map}((E_n)_{n \geq 0}, (F_n)_{n \geq 0})$$

where we omit the subscript in each $\mathrm{Map}(-, -)$.

Next we construct a natural symmetric monoidal functor $\mathrm{Mod}_k^\otimes \rightarrow \mathcal{D}^\otimes(B\Sigma, k)$. For this, let us consider a symmetric monoidal functor $p : \mathrm{Comp}(k) \rightarrow \mathrm{Fun}(B\Sigma^{op}, \mathrm{Comp}(k))$ given by V to $p(V) = (V, 0, 0, \dots)$. It is a left adjoint functor; the right adjoint is determined by evaluation at the 0-th term $(E_0, E_1, \dots) \mapsto E_0$. There is a combinatorial symmetric monoidal model structure on $\mathrm{Comp}(k)$ on which (i) weak equivalences are quasi-isomorphisms, (ii) cofibrations are degreewise monomorphisms and (iii) fibrations are degreewise epimorphisms (cf. [31, 7.1.2.8, 7.1.2.11], [21, 2.3.11]). We remark that generating cofibrations in $\mathrm{Comp}(k)$ (described in the proof of [31, 7.1.2.8] or [21, 2.3.3]) map to cofibrations in $\mathrm{Fun}(B\Sigma^{op}, \mathrm{Comp}(k))$. Hence p preserves cofibrations and weak equivalences. In particular, p is a left Quillen adjoint functor. Inverting weak equivalences of full subcategories of cofibrant objects we have a symmetric monoidal colimit-preserving functor

$$\mathrm{N}_W(\mathrm{Comp}(k)^c) \longrightarrow \mathrm{N}_W(\mathrm{Fun}(B\Sigma^{op}, \mathrm{Comp}(k))^c) = \mathcal{D}(B\Sigma, k).$$

Thanks to [31, 7.1.2.13], there is a natural symmetric monoidal equivalence $\mathrm{N}_W(\mathrm{Comp}(k)^c) \simeq \mathrm{Mod}_k$. Hence we obtain a symmetric monoidal colimit-preserving functor $\mathrm{Mod}_k^\otimes \rightarrow \mathcal{D}^\otimes(B\Sigma, k)$. Thus $\mathrm{Mod}_k^\otimes \rightarrow \mathcal{D}^\otimes(B\Sigma, k)$ belongs to $\mathrm{CAlg}(\mathrm{Pr}_k^L) \simeq \mathrm{CAlg}(\mathrm{Pr}_k^L)_{\mathrm{Mod}_k^\otimes /}$.

Lemma 3.4. *Let S be a small ∞ -category. Let R be a commutative ring spectrum. Let*

$$S \rightarrow \mathrm{Fun}(S^{op}, S) \rightarrow \mathrm{Fun}(S^{op}, S_*) \rightarrow \mathrm{Fun}(S^{op}, \mathrm{Sp}) \rightarrow \mathrm{Fun}(S^{op}, \mathrm{Mod}_R)$$

be the sequence of functors; the first functor is the Yoneda embedding, the other functors are determined by the composition with $S \rightarrow S_ \xrightarrow{\Sigma^\infty} \mathrm{Sp} \xrightarrow{\otimes_S^R} \mathrm{Mod}_R$ where $S \rightarrow S_*$ carries A to $A \sqcup \Delta^0$. Let \mathcal{C} be an R -linear presentable ∞ -category, that is, an object in Pr_R^L . Then $\mathrm{Fun}(S^{op}, \mathrm{Mod}_R) \simeq \mathrm{Fun}(S^{op}, S) \otimes \mathrm{Mod}_R$, and the composition with the composite $S \rightarrow \mathrm{Fun}(S^{op}, \mathrm{Mod}_R)$ induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{Pr}_R^L}(\mathrm{Fun}(S^{op}, \mathrm{Mod}_R), \mathcal{C}) \rightarrow \mathrm{Map}(S, \mathcal{C}).$$

Proof. By the left Kan extension (cf. [30, 5.1.5.6]), the Yoneda embedding induces

$$\mathrm{Map}_{\mathrm{Pr}^L}(\mathrm{Fun}(S^{op}, S), \mathcal{C}) \simeq \mathrm{Map}(S, \mathcal{C})$$

for any $\mathcal{C} \in \mathrm{Pr}^L$. Consider the adjoint pair $\mathrm{Pr}^L \rightleftarrows \mathrm{Pr}_R^L$ where the right adjoint $\mathrm{Pr}_R^L \rightarrow \mathrm{Pr}^L$ is the forgetful functor, and the left adjoint is given by the base change $(-) \otimes \mathrm{Mod}_R$. Taking account of this adjoint pair $\mathrm{Pr}^L \rightleftarrows \mathrm{Pr}_R^L$ we have

$$\mathrm{Map}_{\mathrm{Pr}^L}(\mathrm{Fun}(S^{op}, S), \mathcal{C}) \simeq \mathrm{Map}_{\mathrm{Pr}_R^L}(\mathrm{Fun}(S^{op}, S) \otimes \mathrm{Mod}_R, \mathcal{C})$$

for any $\mathcal{C} \in \mathrm{Pr}_R^L$. Next we show that $\mathrm{Fun}(S^{op}, S) \otimes \mathcal{D} \simeq \mathrm{Fun}(S^{op}, \mathcal{D})$ for any $\mathcal{D} \in \mathrm{Pr}^L$. By the definition

$$\mathrm{Fun}(S^{op}, S) \otimes \mathcal{D} \simeq \mathrm{Fun}^R(\mathcal{D}^{op}, \mathrm{Fun}(S^{op}, S)) \simeq \mathrm{Fun}'(\mathcal{D}^{op} \times S^{op}, S)$$

where $\mathrm{Fun}'(\mathcal{D}^{op} \times S^{op}, S)$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{D}^{op} \times S^{op}, S)$ spanned by functors which preserves limits in the variable \mathcal{D}^{op} . There exist equivalences

$$\mathrm{Fun}'(\mathcal{D}^{op} \times S^{op}, S) \simeq \mathrm{Fun}(S^{op}, \mathrm{Fun}^R(\mathcal{D}^{op}, S)) \simeq \mathrm{Fun}(S^{op}, \mathcal{D} \otimes S) \simeq \mathrm{Fun}(S^{op}, \mathcal{D}).$$

Thus $S \rightarrow \mathrm{Fun}(S^{op}, \mathrm{Mod}_R)$ induces the desired equivalence. \square

Lemma 3.5. *There is a natural equivalence*

$$\mathrm{Fun}(B\Sigma^{op}, \mathrm{Mod}_R) \simeq \bigoplus_{n \geq 0} \mathrm{Fun}(B\Sigma_n^{op}, \mathrm{Mod}_R)$$

in Pr_R^L . Here the coproduct $\bigoplus_{n \geq 0}$ of the right hand side is taken in Pr_R^L .

Proof. Invoking Lemma 3.4, we have

$$\begin{aligned} \mathrm{Map}_{\mathrm{Pr}_R^L}(\mathrm{Fun}(B\Sigma^{op}, \mathrm{Mod}_R), \mathcal{C}) &\simeq \mathrm{Map}(B\Sigma, \mathcal{C}) \\ &\simeq \prod_{n \geq 0} \mathrm{Map}(B\Sigma_n, \mathcal{C}) \\ &\simeq \prod_{n \geq 0} \mathrm{Map}_{\mathrm{Pr}_R^L}(\mathrm{Fun}(B\Sigma_n^{op}, \mathrm{Mod}_R), \mathcal{C}) \\ &\simeq \mathrm{Map}_{\mathrm{Pr}_R^L}(\bigoplus_{n \geq 0} \mathrm{Fun}(B\Sigma_n^{op}, \mathrm{Mod}_R), \mathcal{C}) \end{aligned}$$

for any $\mathcal{C} \in \mathrm{Pr}_R^L$. This proves our assertion. \square

Remark 3.6. The ∞ -category $\mathrm{Fun}(B\Sigma^{op}, \mathrm{Mod}_R)$ is not a coproduct of $\{\mathrm{Fun}(B\Sigma_n^{op}, \mathrm{Mod}_R)\}_{n \geq 0}$ in $\widehat{\mathrm{Cat}}_\infty$.

Proposition 3.7. *Suppose that \mathcal{C}^\otimes belongs to $\mathrm{CAlg}(\mathrm{Pr}_k^L)$. There exists a natural equivalence*

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}_k^L)}(\mathcal{D}^\otimes(B\Sigma, k), \mathcal{C}^\otimes) \simeq \mathcal{C}^\otimes.$$

To prove Proposition 3.7, we first recall the notion of free commutative algebra objects in (general) symmetric monoidal ∞ -categories (cf. [31, 3.1]). Let \mathcal{C}^\otimes be a symmetric monoidal ∞ -category and $\mathrm{CAlg}(\mathcal{C})$ the ∞ -category of commutative algebra objects. We denote by $\theta : \mathrm{CAlg}(\mathcal{C}) \rightarrow \mathcal{C}$ the forgetful functor. For $C \in \mathcal{C}$, $A \in \mathrm{CAlg}(\mathcal{C})$ and $\phi : C \rightarrow \theta(A)$, we say that ϕ makes A a free commutative algebra object generated by C if $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(A, B) \rightarrow \mathrm{Map}_{\mathcal{C}}(C, \theta(B))$, informally given by $f \mapsto \theta(f) \circ \phi$, is a homotopy equivalence. If we suppose that \mathcal{C} admits countable colimits and the tensor product preserves countable colimits separately in each variable, then θ has a left adjoint $\mathrm{Free}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathrm{CAlg}(\mathcal{C})$, so that $(\mathrm{Free}_{\mathcal{C}}(C), C \rightarrow \mathrm{Free}_{\mathcal{C}}(C))$ is a free commutative algebra object generated by C where $C \rightarrow \mathrm{Free}_{\mathcal{C}}(C)$ is the unit map determined by the adjoint pair.

Consider the free commutative symmetric monoidal ∞ -category $\mathrm{Free}(\Delta^0)$ generated by the “trivial” category Δ^0 . More precisely, $\mathrm{Free}(\Delta^0)$ is the image of Δ^0 under the left adjoint functor Free in

$$\mathrm{Free} : \mathrm{Cat}_\infty \rightleftarrows \mathrm{CAlg}(\mathrm{Cat}_\infty) : \theta = \text{forget}.$$

The free algebra object $\mathrm{Free}(\Delta^0)$ has a more explicit form $B\Sigma$. We define a (strict) symmetric monoidal structure on $B\Sigma$. The tensor product $\otimes : B\Sigma \times B\Sigma \rightarrow B\Sigma$ is given by $\bar{n} \otimes \bar{m} := \overline{n + m}$. A pair of maps $\phi : \bar{n} \rightarrow \bar{n}$ and $\psi : \bar{m} \rightarrow \bar{m}$ induces the map $\phi \otimes \psi : \overline{n + m} \rightarrow \overline{n + m}$ determined by the permutations of $\{1, \dots, n\}$ and $\{n + 1, \dots, n + m\}$ given by ϕ and ψ respectively. The commutative constraint $\bar{n} \otimes \bar{m} = \overline{n + m} \rightarrow \overline{n + m} = \bar{m} \otimes \bar{n}$ is given by the left multiplication by the permutation $(1, \dots, n, n + 1, \dots, n + m) \mapsto (n + 1, \dots, n + m, 1, \dots, n)$. The unit object is $\bar{0}$.

Proposition 3.8. *Let $v : \Delta^0 \rightarrow B\Sigma$ be a functor determined by the value $\bar{1}$. Then a pair $(B\Sigma, v : \Delta^0 \rightarrow B\Sigma)$ is a free commutative algebra object in Cat_∞ , generated by Δ^0 . In particular, there exists a symmetric monoidal equivalence $\mathrm{Free}(\Delta^0) \simeq B\Sigma$.*

Before giving the proof of Proposition 3.8, we recall $\mathrm{Sym}^*(-)$ from [32, III Section 3, arXiv:math/0703204v3] (see also [31, 3.1.3] for a more general setting, but we here refer to the simple version described in the arXiv version). Let Fin_*^\sim be the subcategory of Fin_* such

that (i) objects in Fin_*^\sim are same with Fin_* , and (ii) a morphism of Fin_* lies in Fin_*^\sim if and only if it is an isomorphism. Notice that $N(\text{Fin}_*^\sim) \simeq B\Sigma$. For an ∞ -category \mathcal{C} we refer to a functor $N(\text{Fin}_*^\sim) \rightarrow \mathcal{C}$ as a symmetric sequence in \mathcal{C} . Roughly speaking, a symmetric sequence in \mathcal{C} consists of data $\{C_n\}_{n \geq 0}$ where each C_n is endowed with the left action of Σ_n . As constructed in [32, III Section 3] for any symmetric monoidal ∞ -category there is a functor $\text{PSym} : \mathcal{C} \rightarrow \text{Fun}(N(\text{Fin}_*^\sim), \mathcal{C})$ which sends C to $\{C^{\otimes n}\}_{n \geq 0}$ such that each $C^{\otimes n}$ is equipped with the permutation action of Σ_n . Suppose that \mathcal{C} has countable colimits. We define $\text{Sym}^* : \mathcal{C} \rightarrow \mathcal{C}$ to be the composite

$$\mathcal{C} \xrightarrow{\text{PSym}} \text{Fun}(N(\text{Fin}_*^\sim), \mathcal{C}) \rightarrow \mathcal{C}$$

the right functor carries diagrams to their colimits. If $\text{Fin}_*^\sim(n)$ is the full subcategory of Fin_*^\sim spanned by $\langle n \rangle$, then we define Sym^n to be the composite

$$\mathcal{C} \xrightarrow{\text{PSym}} \text{Fun}(N(\text{Fin}_*^\sim), \mathcal{C}) \rightarrow \text{Fun}(N(\text{Fin}_*^\sim(n)), \mathcal{C}) \rightarrow \mathcal{C}$$

where the middle functor is induced by the restriction and the right functor carries diagrams to colimits. By the definition $\text{Sym}^n C$ is the colimit of the permutation action of Σ_n on $C^{\otimes n}$. The $\text{Sym}^* C$ is the coproduct $\sqcup_{n \geq 0} \text{Sym}^n C$.

Proof. We apply [32, III, 3.12] to our situation: $B\Sigma$ is a free commutative algebra object generated by Δ^0 if and only if the composite $\text{Sym}^*(\Delta^0) \xrightarrow{\text{Sym}(v)} \text{Sym}^*(B\Sigma) \rightarrow B\Sigma$ is a categorical equivalence. Here $\text{Sym}^*(B\Sigma) \simeq \sqcup_{n \geq 0} \text{Sym}^n(B\Sigma) \rightarrow B\Sigma$ is induced by the evaluation of the natural transformation $\text{PSym}(B\Sigma) \rightarrow B\Sigma_{N(\text{Fin}_*^\sim)}$ from $\text{PSym}(B\Sigma) : N(\text{Fin}_*^\sim) \rightarrow \text{Cat}_\infty$ to the constant functor $B\Sigma_{N(\text{Fin}_*^\sim)} : N(\text{Fin}_*^\sim) \rightarrow \text{Cat}_\infty$ taking the value $B\Sigma$ (see [32, III, 3.10]). In concrete terms, for each $n \geq 0$ the evaluation at $\langle n \rangle$ induces the n -fold tensor product $B\Sigma^{\times n} \rightarrow B\Sigma$ which factors through $\text{Sym}^n(B\Sigma) \rightarrow B\Sigma$. To prove that the composite is an equivalence, note first that

$$B\Sigma^{\times n} = \sqcup_{(r_1, \dots, r_n)} B\Sigma_{r_1} \times \dots \times B\Sigma_{r_n}$$

since the cartesian product in Cat_∞ preserves colimits separately in each variable. Hence $B\Sigma_1 \times \dots \times B\Sigma_1$ is a direct summand of $B\Sigma^{\times n}$ which is compatible with the permutation (left) action of Σ_n . Note that the action of Σ_n on $B\Sigma_1 \times \dots \times B\Sigma_1$ is trivial since $B\Sigma_1$ is contractible. We have the following diagram:

$$\begin{array}{ccccc} (\Delta^0)^{\times n} & \xrightarrow{\sim} & B\Sigma_1^{\times n} & \xrightarrow{f} & B\Sigma_n \\ \downarrow & & \downarrow & \nearrow g & \\ \text{Sym}^n(\Delta^0) = B\Sigma_n & \xrightarrow{\sim} & B\Sigma_1^{\times n} / \Sigma_n & & \end{array}$$

The vertical functors are natural projections. The functor f is induced by the n -fold tensor product $B\Sigma^{\times n} \rightarrow B\Sigma$. By the commutative constraint of the symmetric monoidal structure of $B\Sigma$, f factors through the projection $B\Sigma_1^{\times n} \rightarrow B\Sigma_1^{\times n} / \Sigma_n$, which gives rise to g . Here we consider $B\Sigma_1^{\times n} / \Sigma_n$ as a direct summand of $\text{Sym}^n(B\Sigma)$, and g is $B\Sigma_1^{\times n} / \Sigma_n \hookrightarrow \text{Sym}^n(B\Sigma) \rightarrow B\Sigma$. The lower horizontal functor is induced by $\text{Sym}^*(v)$. It will suffice to show that g is a categorical equivalence. The functor g is determined by f . More precisely, we think of f as the morphism in $\text{Fun}(B\Sigma_n, \text{Cat}_\infty)$, i.e., the natural transformation from the constant functor $B\Sigma_n \rightarrow \text{Cat}_\infty$ taking the value $B\Sigma_1^{\times n}$ to the constant functor $B\Sigma_n \rightarrow \text{Cat}_\infty$ taking the value $B\Sigma_n$. Note that for any group G there is an adjoint pair

$$\alpha : \text{Fun}(BG, \text{Cat}_\infty) \rightleftarrows \text{Cat}_\infty : \delta$$

where the right adjoint $\delta : \text{Cat}_\infty \rightarrow \text{Fun}(BG, \text{Cat}_\infty)$ is the diagonal embedding by the composition with $BG \rightarrow \Delta^0$. The left adjoint carries $BG \rightarrow \text{Cat}_\infty$ to its colimit. Through this adjoint pair the morphism f in $\text{Fun}(B\Sigma_n, \text{Cat}_\infty)$ corresponds to $g : B\Sigma_n \rightarrow B\Sigma_n$. In concrete terms,

the data of a functor $h : BG \rightarrow B\Sigma_n$ amounts to a left action of $G = \text{Hom}_{BG}(*_{BG}, *_{BG})$ on $\text{Hom}_{B\Sigma_n}(*_{B\Sigma_n}, *_{B\Sigma_n})$ in the obvious way, where $*_{BG}$ and $*_{B\Sigma_n}$ denote unique objects in BG and $B\Sigma_n$ respectively (keep in mind the case $G = \Sigma_n$). A left action $G = \text{Hom}_{BG}(*_{BG}, *_{BG})$ on $\text{Hom}_{B\Sigma_n}(*_{B\Sigma_n}, *_{B\Sigma_n})$ corresponds to a natural transformation from the constant functor $BG \rightarrow \text{Cat}_\infty$ taking the value Δ^0 to the constant functor taking value $B\Sigma_n$. It relates g with f . The identity functor $B\Sigma_n \rightarrow B\Sigma_n$ corresponds to the natural left multiplication Σ_n on $\Sigma_n = \text{Hom}_{B\Sigma_n}(*_{B\Sigma_n}, *_{B\Sigma_n})$. Therefore it is enough to prove that f corresponds to the natural left multiplication Σ_n on $\Sigma_n = \text{Hom}_{B\Sigma_n}(*_{B\Sigma_n}, *_{B\Sigma_n})$. Recall that f is induced by the n -fold tensor product of $B\Sigma$. By the definition of the commutative constraint of $B\Sigma$, the (trivial) permutation action of Σ_n on $(B\Sigma_1)^{\times n}$ gives rise to the left multiplication of Σ_n on $\Sigma_n = \text{Hom}_{B\Sigma_n}(*_{B\Sigma_n}, *_{B\Sigma_n})$ (consider the natural transformations given by the commutative constraint

$$\begin{array}{ccc} B\Sigma_1^{\times n} & \xrightarrow{\text{trivial action of } \sigma \in \Sigma_n} & B\Sigma_1^{\times n} \\ & \searrow f \quad \swarrow f & \\ & B\Sigma_n & \end{array}$$

which give rise to the action of Σ_n on $\text{Hom}_{B\Sigma_n}(*_{B\Sigma_n}, *_{B\Sigma_n})$). Hence we conclude that g is the identity. \square

Consider the presentable ∞ -category $\text{Fun}(B\Sigma^{op}, \mathcal{S})$. According to [31, 4.8.1.10, 4.8.1.12] $\text{Fun}(B\Sigma^{op}, \mathcal{S})$ inherits from $B\Sigma$ a symmetric monoidal structure with the following properties:

- the Yoneda embedding $B\Sigma \hookrightarrow \text{Fun}(B\Sigma^{op}, \mathcal{S})$ is extended to a symmetric monoidal functor,
- the tensor product $\otimes : \text{Fun}(B\Sigma^{op}, \mathcal{S}) \times \text{Fun}(B\Sigma^{op}, \mathcal{S}) \rightarrow \text{Fun}(B\Sigma^{op}, \mathcal{S})$ preserves small colimits separately in each variable.

Hence $\text{Fun}(B\Sigma^{op}, \mathcal{S})$ belongs to $\text{CAlg}(\text{Pr}^L)$, and let us consider the coproduct

$$\text{Fun}(B\Sigma^{op}, \mathcal{S}) \otimes \text{Mod}_R^\otimes$$

in $\text{CAlg}(\text{Pr}^L)$ for a commutative ring spectrum R . Namely, $\text{Fun}(B\Sigma^{op}, \mathcal{S}) \otimes \text{Mod}_R^\otimes$ lies in $\text{CAlg}(\text{Pr}_R^L)$.

Proposition 3.9. *The sequence of functors $\Delta^0 \xrightarrow{v} B\Sigma \hookrightarrow \text{Fun}(B\Sigma^{op}, \mathcal{S}) \rightarrow \text{Fun}(B\Sigma^{op}, \mathcal{S}) \otimes \text{Mod}_R^\otimes$ induces*

$$\text{Map}_{\text{CAlg}(\text{Pr}_R^L)}(\text{Fun}(B\Sigma^{op}, \mathcal{S}) \otimes \text{Mod}_R^\otimes, \mathcal{C}^\otimes) \xrightarrow{\sim} \text{Map}(\Delta^0, \mathcal{C}) = \mathcal{C}^\simeq$$

for any $\mathcal{C}^\otimes \in \text{CAlg}(\text{Pr}_R^L)$.

Proof. We note the three points:

- $B\Sigma \simeq \text{Free}(\Delta^0)$ by Proposition 3.8,
- $\text{Map}_{\text{CAlg}(\text{Pr}^L)}(\text{Fun}(B\Sigma^{op}, \mathcal{S}), \mathcal{C}^\otimes) \simeq \text{Map}_{\text{CAlg}(\widehat{\text{Cat}_\infty})}(B\Sigma, \mathcal{C}^\otimes)$ by [31, 4.8.1.10]
- $\text{Map}_{\text{CAlg}(\text{Pr}_R^L)}(\text{Fun}(B\Sigma^{op}, \mathcal{S}) \otimes \text{Mod}_R^\otimes, \mathcal{C}^\otimes) \simeq \text{Map}_{\text{CAlg}(\text{Pr}^L)}(\text{Fun}(B\Sigma^{op}, \mathcal{S}), \mathcal{C}^\otimes)$ by the adjoint pair $(-) \otimes \text{Mod}_R^\otimes : \text{CAlg}(\text{Pr}^L) \rightleftarrows \text{CAlg}(\text{Pr}_R^L) : \text{forget}$.

Our claim follows. \square

Suppose that R is the Eilenberg-MacLane spectrum Hk of the field k of characteristic zero. We relate $\text{Fun}(B\Sigma^{op}, \mathcal{S}) \otimes \text{Mod}_R^\otimes$ with $\mathcal{D}(B\Sigma, k)$.

Proposition 3.10. *There exists an equivalence $\text{Fun}(B\Sigma^{op}, \mathcal{S}) \otimes \text{Mod}_k^\otimes \simeq \mathcal{D}^\otimes(B\Sigma, k)$ in $\text{CAlg}(\text{Pr}_k^L)$.*

Proof. Let $B\Sigma \rightarrow \text{Fun}(B\Sigma^{op}, \mathcal{S}) \otimes \text{Mod}_k^\otimes$ be the symmetric monoidal functor given by

$$B\Sigma \hookrightarrow \text{Fun}(B\Sigma^{op}, \mathcal{S}) \rightarrow \text{Fun}(B\Sigma^{op}, \mathcal{S}_*) \rightarrow \text{Fun}(B\Sigma^{op}, \text{Sp}) \rightarrow \text{Fun}(B\Sigma^{op}, \text{Sp}) \otimes \text{Mod}_k$$

where the first functor is the Yoneda embedding, the subsequent functors are given by compositions with $\mathcal{S} \rightarrow \mathcal{S}_* \xrightarrow{\Sigma^\infty} \mathbf{Sp} = \mathbf{Mod}_{\mathcal{S}} \rightarrow \mathbf{Mod}_k$. If we identify $\mathrm{Fun}(B\Sigma^{op}, \mathbf{Sp}) \otimes \mathbf{Mod}_k$ with $\mathrm{Fun}(B\Sigma^{op}, \mathbf{Mod}_k)$ by Lemma 3.4, the image of $\bar{r} \in B\Sigma$ that lies in $\mathrm{Fun}(B\Sigma^{op}, \mathbf{Mod}_k) \simeq \prod_{n \geq 0} \mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k)$ is $J^r := (J_n^r)_{n \geq 0}$ such that $J_n^r \in \mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k)$, $J_r^r = \bigoplus_{g \in \Sigma_r} k \cdot g = k[\Sigma_r] \in \mathbf{Mod}_k$ equipped with the right multiplication of Σ_r , and $J_n^r = 0$ for $n \neq r$. Here we regard J_n^r as an object in \mathbf{Mod}_k endowed with right action of Σ_n (arising from the functoriality of $B\Sigma_n^{op} \rightarrow \mathbf{Mod}_k$).

By Proposition 3.9 the object I^1 in $\mathcal{D}(B\Sigma, k)$ induces a morphism $\phi : \mathrm{Fun}(B\Sigma^{op}, \mathcal{S}) \otimes \mathbf{Mod}_k^\otimes \rightarrow \mathcal{D}^\otimes(B\Sigma, k)$ in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$. To prove that it is a symmetric monoidal equivalence, it will suffice to show that ϕ induces a categorical equivalence of underlying ∞ -categories. Since $\mathcal{D}(B\Sigma, k)$ and $\mathrm{Fun}(B\Sigma^{op}, \mathbf{Mod}_k)$ are stable, and $\mathrm{Fun}(B\Sigma^{op}, \mathbf{Mod}_k) \rightarrow \mathcal{D}(B\Sigma, k)$ is exact, thus it is enough to prove that ϕ induces an equivalence between their homotopy categories (see e.g. [24, Lemma 5.8]). Since J^1 maps to I^1 , thus $(J^1)^{\otimes r} = J^r$ maps to $(I^1)^{\otimes r} = I^r$. Thus the colimit-preserving functor

$$\phi : \mathrm{Fun}(B\Sigma^{op}, \mathbf{Mod}_k) \simeq \prod_{n \geq 0} \mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k) \rightarrow \mathcal{D}(B\Sigma, k) \simeq \prod_{n \geq 0} \mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k)$$

is determined by the product of each restriction $\phi_n : \mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k) \rightarrow \mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k)$. Here $\mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k)$ is considered as the full subcategory of $\prod_{n \geq 0} \mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k)$ spanned by $(E_i)_{i \geq 0}$ such that $E_i = 0$ for $n \neq i$. Thus it will suffice to prove that ϕ_n induces an equivalence of homotopy categories. To this end, consider the map

$$\theta : \mathrm{Hom}_{\mathrm{h}(\mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k))}(k[\Sigma_n], k[\Sigma_n]) \rightarrow \mathrm{Hom}_{\mathrm{h}(\mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k))}(k[\Sigma_n], k[\Sigma_n])$$

induced by $\mathrm{h}(\phi_n) : \mathrm{h}(\mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k)) \rightarrow \mathrm{h}(\mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k))$. Recall that $\mathrm{h}(\mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k))$ is the (unbounded) derived category of k -linear representations of Σ_n . Note that the category of k -linear representations of Σ_n is semi-simple, and every irreducible representation of Σ_n is isomorphic to a direct summand of $k[\Sigma_n]$. Therefore, to show that the exact functor $\mathrm{h}(\phi_n)$ of triangulated categories is an equivalence, we are reduced to proving that θ is a bijective map. Observe that $\mathrm{Hom}_{\mathrm{h}(\mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k))}(k[\Sigma_n], k[\Sigma_n])$ can be identified with the set of homomorphisms $k[\Sigma_n] \rightarrow k[\Sigma_n]$ as right $k[\Sigma_n]$ -modules. Thus it is isomorphic to $k[\Sigma_n]$, and we can view θ as a k -linear morphism $\xi : k[\Sigma_n] \rightarrow k[\Sigma_n]$. By the construction of ϕ , $\mathrm{h}(\phi_n)$ commutes with the natural functor $B\Sigma_n \rightarrow \mathrm{h}(\mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k))$. Hence the k -linear map $\xi : k[\Sigma_n] \rightarrow k[\Sigma_n]$ preserves $\Sigma_n \subset k[\Sigma_n]$. It follows that θ is a bijective map. \square

Proof of Proposition 3.7. It follows from Proposition 3.9 and 3.10. \square

Let K be the standard representation of GL_d , that is, $k^{\oplus d}$ endowed with the natural action of GL_d . Applying Proposition 3.7 to K we obtain a morphism in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$:

$$u : \mathcal{D}^\otimes(B\Sigma, k) \rightarrow \mathcal{D}^\otimes(B\mathrm{GL}_d) \simeq \mathrm{Rep}^\otimes(\mathrm{GL}_d) = \mathrm{QC}^\otimes(B\mathrm{GL}_d)$$

which carries I^1 to K placed in degree zero. Since $I^n = (I^1)^{\otimes n}$, thus $u(I^n) = K^{\otimes n}$. Moreover, we have

Proposition 3.11. *Suppose that W is a representation of Σ_n which is viewed as an object in $\mathrm{Fun}(B\Sigma_n^{op}, \mathbf{Mod}_k) \subset \mathcal{D}(B\Sigma, k)$. Then $u(W) \simeq W \otimes_{k[\Sigma_n]} K^{\otimes n}$.*

Proof. Note first that W can be described as a coproduct of retracts in $k[\Sigma_n]$. Thus we may and will assume that W is a retract of $k[\Sigma_n]$. Since W is a retract, it is a filtered colimit of the linearly ordered sequence consisting of the idempotent maps (the standard heart consisting of part of (co)homological degree zero is closed under formulation of filtered colimits), and u preserves small colimits, thus $u(W)$ is a filtered colimit of the linearly ordered sequence of idempotent maps between $u(I^n) = K^{\otimes n} \simeq K[\Sigma_n] \otimes_{K[\Sigma_n]} K^{\otimes n}$. The standard heart of $\mathcal{D}(B\mathrm{GL}_d)$ is also closed under filtered colimits. Thus we conclude that $u(W) \simeq W \otimes_{k[\Sigma_n]} K^{\otimes n}$. \square

Before proceeding further we need the representation theory of symmetric groups. Let us recall that every representation of a symmetric group can be constructed by means of Young diagrams (see e.g. [12, 4.1], [13, Section 7]): Let λ be a Young diagram having n boxes. Then after choosing a standard Young tableau whose underlying Young diagram is λ , we can associate to it an idempotent map between $k[\Sigma_n]$ called the Young symmetrizer. Its retract V_λ (the image of the idempotent map) is an irreducible representation of Σ_n . The isomorphism class of V_λ (as a representation) does not depend on the choice of a Young tableau. Any irreducible representation of a symmetric group is obtained in this way for a unique Young diagram. The tensor product of irreducible representations can be described by the Littlewood-Richardson rule.

Let us consider any k -linear representation of Σ_n for $n \geq 0$ as an object in $\text{Fun}(B\Sigma_n^{op}, \text{Mod}_k) \subset \mathcal{D}(B\Sigma, k)$. Let T be the set consisting of objects W in $\mathcal{D}(B\Sigma, k)$ such that W is of the form $V[r]$ such that $[r]$ indicates the shift for $r \in \mathbb{Z}$, V is an irreducible representation of some Σ_n associated to Young diagrams having more than d rows.

Lemma 3.12. *There are an object $\mathcal{D}^\otimes(B\Sigma, k)_T$ and a morphism $\mathcal{D}^\otimes(B\Sigma, k) \rightarrow \mathcal{D}^\otimes(B\Sigma, k)_T$ in $\text{CAlg}(\text{Pr}_k^L)$ such that the composition induces a homotopy equivalence of spaces*

$$\text{Map}_{\text{CAlg}(\text{Pr}_k^L)}(\mathcal{D}^\otimes(B\Sigma, k)_T, \mathcal{C}^\otimes) \rightarrow \text{Map}_{\text{CAlg}(\text{Pr}_k^L)}^\wedge(\mathcal{D}^\otimes(B\Sigma, k), \mathcal{C}^\otimes)$$

for any $\mathcal{C}^\otimes \in \text{CAlg}(\text{Pr}_k^L)$. Here the superscript \wedge indicates that we consider only those functors which carry the $(d+1)$ -fold wedge product $\wedge^{d+1}(I^1)$ to zero.

Proof. Notice first that for any $\text{Mod}_k^\otimes \rightarrow \mathcal{E}^\otimes \in \text{CAlg}(\text{Pr}^L)_{\text{Mod}_k^\otimes} \simeq \text{CAlg}(\text{Pr}_k^L)$

$$\text{Map}_{\text{CAlg}(\text{Pr}_k^L)}(\mathcal{E}^\otimes, \mathcal{C}^\otimes) \simeq \text{Map}_{\text{CAlg}(\text{Pr}^L)}(\mathcal{E}^\otimes, \mathcal{C}^\otimes) \times_{\text{Map}_{\text{CAlg}(\text{Pr}^L)}(\text{Mod}_k^\otimes, \mathcal{C}^\otimes)} \{s\}$$

where the right hand side denotes the homotopy limit, and $s : \text{Mod}_k^\otimes \rightarrow \mathcal{C}^\otimes$ is the structure functor of $\mathcal{C}^\otimes \in \text{CAlg}(\text{Pr}_k^L)_{\text{Mod}_k^\otimes}$. Therefore we may replace $\text{CAlg}(\text{Pr}_k^L)$ by $\text{CAlg}(\text{Pr}^L)$ in the statement. We apply symmetric monoidal localizations [31, 4.1.3.4] to $T' := \{W \rightarrow 0\}_{W \in T}$. For this, we need to show that for any $W \in T$ and any $C \in \mathcal{D}(B\Sigma, k)$, $W \otimes C$ is a coproduct of objects in T (it follows that $W \otimes C \rightarrow 0$ belongs to a strongly saturated class generated by the small set T' ; cf. [30, 5.5.4.5]). We deduce it from Littlewood-Richardson rule (or its special case: Pieri rule); [13, Section 5]. For this purpose, we may assume that W is an irreducible representation V_λ associated to a Young diagram λ having m rows with $m > d$, and C is an irreducible representations V_μ associated to a Young diagram μ . Let $\alpha = (1, \dots, 1)$ be the Young diagram corresponding to the partition $m = 1 + \dots + 1$ of m , that is, α has m boxes in one column. Let $\lambda - \alpha$ be the Young diagram obtained from λ by removing m boxes from the left end column. Then by Littlewood-Richardson rule we see that the decomposition in $\mathcal{D}(B\Sigma, k)$

$$V_\alpha \otimes V_{\lambda-\alpha} \simeq \oplus_\nu V_\nu$$

where the right hand side is a coproduct of those V_ν such that Young diagram ν is obtained from $\lambda - \alpha$ by adding m boxes, with no two in the same row. Hence V_λ is a retract of $V_\alpha \otimes V_{\lambda-\alpha}$. Thus it is enough to prove that $V_\alpha \otimes V_{\lambda-\alpha} \otimes V_\mu$ is decomposed into a coproduct of the representations V_β such that β has more than d rows. For this, we may replace $V_{\lambda-\alpha} \otimes V_\mu$ by V_μ . Then again by Littlewood-Richardson rule we see that $V_\alpha \otimes V_\mu$ is decomposed into $\oplus_\beta V_\beta$ where β run over the set of Young diagrams obtained from μ by adding m boxes, with no two in the same row. In particular, β has at least m rows. Consequently, we can apply symmetric monoidal localization [31, 4.1.3.4] with respect to T' ; inverting T' we obtain $\mathcal{D}^\otimes(B\Sigma, k) \rightarrow \mathcal{D}^\otimes(B\Sigma, k)_T := \mathcal{D}^\otimes(B\Sigma, k)[T'^{-1}]$ which induces a homotopy equivalence

$$\text{Map}_{\text{CAlg}(\text{Pr}^L)}(\mathcal{D}^\otimes(B\Sigma, k)_T, \mathcal{C}^\otimes) \rightarrow \text{Map}_{\text{CAlg}(\text{Pr}^L)}^T(\mathcal{D}^\otimes(B\Sigma, k), \mathcal{C}^\otimes)$$

where the superscript T indicates that we consider only those functors which carry all objects in T to zero. Finally, we prove that any morphism $F : \mathcal{D}^\otimes(B\Sigma, k) \rightarrow \mathcal{C}^\otimes$ in $\text{CAlg}(\text{Pr}^L)$ sends all

objects in T to zero if and only if it sends $\wedge^{d+1}(I^1)$ to zero. The “only if” direction is obvious since the $(d+1)$ -fold wedge product is obtained from $k[\Sigma_{d+1}] \simeq (I^1)^{\otimes d+1}$ by using the Young symmetrizer arising from the Young diagram having $d+1$ boxes in one column. Suppose that F sends $\wedge^{d+1}(I^1)$ to zero. As observed above, if the Young diagram λ has m rows with $m > d$, then $V_\lambda \in \mathcal{D}(B\Sigma, k)$ is a retract of a tensor product of $\wedge^m(I^1)$ and another object. Therefore V_λ maps to zero. \square

Remark 3.13. The underlying functor $\mathcal{D}(B\Sigma, k) \rightarrow \mathcal{D}(B\Sigma, k)_T$ is a localization (cf. [30, 5.2.7.2]), i.e., a left adjoint functor which has a fully faithful right adjoint functor whose essential image consists of T' -local objects. It sends C to a T' -local object C_T such that the unit map $C \rightarrow C_T$ is a T' -equivalence (cf. [30, 5.2.7, 5.5.4.1, 5.5.4.15]). Suppose that C is $\oplus_{i \in I} M_i$ of a coproduct of those M_i such that M_i is of the form $N[r]$ where N is an irreducible representation of some Σ_m and $r \in \mathbb{Z}$. Then C_T is isomorphic to the retract of $\oplus_{i \in I} M_i$ obtained by removing retracts belonging to T .

For an irreducible representation V_λ of Σ_n associated to a Young diagram λ , $V_\lambda \otimes_{k[\Sigma_n]} K^{\otimes n}$ is zero if and only if the number of rows of λ is bigger than d . By Proposition 3.11, we see that $u(W) \simeq 0$ for any $W \in T$. Hence invoking Lemma 3.12 we obtain a morphism $u_T : \mathcal{D}^\otimes(B\Sigma, k)_T \rightarrow \mathcal{D}^\otimes(BGL_d)$ induced by $u : \mathcal{D}^\otimes(B\Sigma, k) \rightarrow \mathcal{D}^\otimes(BGL_d)$. Let $\mathcal{D}(BGL_d)_{\text{eff}}$ be the stable subcategory which contains the standard representation K and the unit and is closed under tensor product and coproducts. The stable presentable full subcategory $\mathcal{D}(BGL_d)_{\text{eff}}$ inherits a symmetric monoidal structure from $\mathcal{D}^\otimes(BGL_d)$.

Proposition 3.14. *The functor $u_T : \mathcal{D}(B\Sigma, k)_T \rightarrow \mathcal{D}(BGL_d)$ is a fully faithful functor whose essential image is $\mathcal{D}(BGL_d)_{\text{eff}}$. In particular, $\mathcal{D}^\otimes(B\Sigma, k)_T \simeq \mathcal{D}^\otimes(BGL_d)_{\text{eff}}$.*

Before the proof, let us recall the consequences from Schur-Weyl duality. Let V_λ be the irreducible representation of Σ_n associated to a Young diagram λ having n boxes. Then if λ has at most d rows, $V_\lambda \otimes_{k[\Sigma_n]} K^{\otimes n}$ is a nonzero irreducible representation of GL_d . If λ has m rows with $m > d$, then $V_\lambda \otimes_{k[\Sigma_n]} K^{\otimes n} \simeq 0$. One can obtain any irreducible representation of GL_d which is a retract of the power $K^{\otimes n}$ in this way for a unique Young diagram.

Proof. We first prove that $u_T : \mathcal{D}^\otimes(B\Sigma, k)_T \rightarrow \mathcal{D}^\otimes(BGL_d)_{\text{eff}}$ is essentially surjective. Note that by the semi-simplicity any object in $\mathcal{D}(BGL_d)_{\text{eff}}$ is isomorphic to a coproduct $\oplus_{i \in I} P_i$ such that P_i is (up to shift) equivalent to an irreducible representation of GL_d which is contained in $K^{\otimes n}$ for some $n \geq 0$ as a retract. For any nonzero irreducible representation W of GL_d contained in $K^{\otimes n}$, there is a unique irreducible representation V of Σ_n , up to isomorphisms, such that $V \otimes_{k[\Sigma_n]} K^{\otimes n} \simeq W$. Thus Proposition 3.11 implies that u_T is essentially surjective. Next we will prove that $u_T : \mathcal{D}^\otimes(B\Sigma, k)_T \rightarrow \mathcal{D}^\otimes(BGL_d)_{\text{eff}}$ is fully faithful. Let C and D be objects in $\mathcal{D}(B\Sigma, k)$. We may and will assume that C lies in $\text{Fun}(B\Sigma_n^{\text{op}}, \text{Mod}_k)$ and D lies in $\text{Fun}(B\Sigma_m^{\text{op}}, \text{Mod}_k)$. Suppose that $n \neq m$. Then $\text{Map}_{\mathcal{D}(B\Sigma, k)}(C, D)$ is a contractible space. On the other hand, if $P[r], Q[s] \in \mathcal{D}(BGL_d)$ such that r and s are integers, and P and Q are retracts in $K^{\otimes n}$ and $K^{\otimes m}$ respectively, then $\text{Map}_{\mathcal{D}(BGL_d)}(P[r], Q[s])$ is a contractible space (by weight reason). It follows that $\Delta^0 \simeq \text{Map}_{\mathcal{D}(B\Sigma, k)}(C, D) \rightarrow \text{Map}_{\mathcal{D}(BGL_d)}(u_T(C), u_T(D)) \simeq \Delta^0$ is a homotopy equivalence. Finally, consider the case of $n = m$. To prove $\text{Map}_{\mathcal{D}(B\Sigma, k)}(C, D) \rightarrow \text{Map}_{\mathcal{D}(BGL_d)}(u_T(C), u_T(D))$ is a homotopy equivalence, using decompositions and shifts we are reduced to the case when C and E are irreducible representations of Σ_n , and $D = E[r]$ for some $r \in \mathbb{Z}$. When $C \simeq E$ and $r \geq 0$, then we have a natural homotopy equivalence $k[r] \simeq \text{Map}_{\mathcal{D}(B\Sigma, k)}(C, D) \rightarrow \text{Map}_{\mathcal{D}(BGL_d)}(u_T(C), u_T(D)) \simeq k[r]$. Here for a space S , by $S \simeq k[r]$ we means that $\pi_r(S) \simeq k$ and $\pi_l(S)$ is trivial for $l \neq r$ (i.e., an Eilenberg-MacLane space). When either C is not equivalent to E or $r < 0$, then both $\text{Map}_{\mathcal{D}(B\Sigma, k)}(C, D)$ and $\text{Map}_{\mathcal{D}(BGL_d)}(u_T(C), u_T(D))$ are contractible. This proves that u_T is fully faithful. \square

Let $\mathcal{C}^\otimes \in \text{CAlg}(\text{Pr}_k^{\text{L}})$ and let C be an object in \mathcal{C} . Then there is a categorical construction which makes C an invertible object, i.e., there is an object C^\vee such that $C \otimes C^\vee$ is a unit of \mathcal{C} . Namely, we say that $\mathcal{C}^\otimes \rightarrow \mathcal{C}^\otimes[C^{-1}]$ in $\text{CAlg}(\text{Pr}_k^{\text{L}})$ is the inversion of C if it induces a homotopy equivalence

$$\text{Map}_{\text{CAlg}(\text{Pr}_k^{\text{L}})}(\mathcal{C}^\otimes[C^{-1}], \mathcal{E}^\otimes) \rightarrow \text{Map}_{\text{CAlg}(\text{Pr}_k^{\text{L}})}^C(\mathcal{C}^\otimes, \mathcal{E}^\otimes)$$

for any $\mathcal{E}^\otimes \in \text{CAlg}(\text{Pr}_k^{\text{L}})$, where the superscript C in the right hand side indicates that we consider only those functors which carries C to an invertible object in \mathcal{E}^\otimes . By a result of Robalo [35, 4.10], there is the inversion of C for any \mathcal{C}^\otimes .

Proposition 3.15. *Let $U := \wedge^d K$ be the d -fold wedge product of the standard representation. Let $\mathcal{D}^\otimes(\text{BGL}_d)_{\text{eff}} \rightarrow \mathcal{D}^\otimes(\text{BGL}_d)_{\text{eff}}[U^{-1}]$ be the inversion of U . Then the natural inclusion $\mathcal{D}^\otimes(\text{BGL}_d)_{\text{eff}} \hookrightarrow \mathcal{D}^\otimes(\text{BGL}_d)_{\text{eff}}[U^{-1}]$ induces an equivalence*

$$\mathcal{D}^\otimes(\text{BGL}_d)_{\text{eff}}[U^{-1}] \rightarrow \mathcal{D}^\otimes(\text{BGL}_d).$$

Proof. Let $\mathcal{D}_c(\text{BGL}_d)_{\text{eff}}$ be the (stable) full subcategory $\mathcal{D}(\text{BGL}_d)_{\text{eff}}$ spanned by compact objects. Namely, $\mathcal{D}_c(\text{BGL}_d)_{\text{eff}}$ consists of objects of those objects M such that M is a finite coproduct $\bigoplus_{i \in I} N_i[r_i]$ where each r_i is an integer, and each N_i is an irreducible representation which belongs to $\mathcal{D}_c(\text{BGL}_d)_{\text{eff}}$. The small stable ∞ -category $\mathcal{D}_c(\text{BGL}_d)_{\text{eff}}$ inherits a symmetric monoidal structure in the natural way. By [35, 4.1., 4.2], there is the “small version” of the inversion of U ; there exist a small symmetric monoidal ∞ -category $\mathcal{D}_c^\otimes(\text{BGL}_d)_{\text{eff}}[U^{-1}]$ and a symmetric monoidal functor $\mathcal{D}_c^\otimes(\text{BGL}_d)_{\text{eff}} \rightarrow \mathcal{D}_c^\otimes(\text{BGL}_d)_{\text{eff}}[U^{-1}]$ which induces a homotopy equivalence

$$\text{Map}_{\text{CAlg}(\text{Cat}_\infty)}(\mathcal{D}_c^\otimes(\text{BGL}_d)_{\text{eff}}[U^{-1}], \mathcal{E}^\otimes) \rightarrow \text{Map}_{\text{CAlg}(\text{Cat}_\infty)}^U(\mathcal{D}_c^\otimes(\text{BGL}_d)_{\text{eff}}, \mathcal{E}^\otimes)$$

for any $\mathcal{E}^\otimes \in \text{CAlg}(\text{Cat}_\infty)$, where the superscript U in the right hand side indicates that we consider only those functors which carry U to an invertible object in \mathcal{E}^\otimes . Then since U is a symmetric object in the sense of [35], by [35, 4.21, 4.24] the underlying ∞ -category $\mathcal{D}_c(\text{BGL}_d)_{\text{eff}}[U^{-1}]$ is equivalent to a colimit of the linearly ordered sequence

$$\mathcal{D}_c(\text{BGL}_d)_{\text{eff}} \xrightarrow{\otimes^U} \mathcal{D}_c(\text{BGL}_d)_{\text{eff}} \xrightarrow{\otimes^U} \mathcal{D}_c(\text{BGL}_d)_{\text{eff}} \xrightarrow{\otimes^U} \dots$$

in $\text{CAlg}(\text{Cat}_\infty)$ (in [35, 4.25], the presentable situation is treated, but the proof is also applicable to this case). In particular, $\mathcal{D}_c(\text{BGL}_d)_{\text{eff}}[U^{-1}]$ is a stable ∞ -category since the filtered colimit of stable ∞ -categories in Cat_∞ is also a stable ∞ -category [31, 1.1.4.6]. Since $(-) \otimes U : \mathcal{D}_c(\text{BGL}_d)_{\text{eff}} \rightarrow \mathcal{D}_c(\text{BGL}_d)_{\text{eff}}$ is a fully faithful exact functor and $(-) \otimes U : \mathcal{D}_c(\text{BGL}_d) \rightarrow \mathcal{D}_c(\text{BGL}_d)$ is an equivalence, the colimit can be identified with the essential image of the natural functor induced by the inclusion $\mathcal{D}_c(\text{BGL}_d)_{\text{eff}} \hookrightarrow \mathcal{D}_c(\text{BGL}_d)$:

$$\begin{aligned} \text{colim}(\mathcal{D}_c(\text{BGL}_d)_{\text{eff}} \xrightarrow{\otimes^U} \dots) &\rightarrow \text{colim}(\mathcal{D}_c(\text{BGL}_d) \xrightarrow{\otimes^U} \mathcal{D}_c(\text{BGL}_d) \xrightarrow{\otimes^U} \dots) \\ &\simeq \mathcal{D}_c(\text{BGL}_d). \end{aligned}$$

Since every object in $\mathcal{D}_c(\text{BGL}_d)$ has the form $(U^\vee)^{\otimes m} \otimes W$ such that $m \in \mathbb{N}$, and W belongs to $\mathcal{D}_c(\text{BGL}_d)_{\text{eff}}$, thus the colimit is $\mathcal{D}_c(\text{BGL}_d)$. Hence we deduce that the natural symmetric monoidal functor $\mathcal{D}_c^\otimes(\text{BGL}_d)_{\text{eff}}[U^{-1}] \rightarrow \mathcal{D}_c^\otimes(\text{BGL}_d)$ is an equivalence. Note that since $(-) \otimes U : \mathcal{D}_c(\text{BGL}_d)_{\text{eff}} \rightarrow \mathcal{D}_c(\text{BGL}_d)_{\text{eff}}$ preserves finite colimits, then for any symmetric monoidal stable ∞ -category \mathcal{E}^\otimes a symmetric monoidal functor $\mathcal{D}_c(\text{BGL}_d)_{\text{eff}}[U^{-1}] \rightarrow \mathcal{E}^\otimes$ preserves finite colimits if and only if the composite $\mathcal{D}_c(\text{BGL}_d)_{\text{eff}} \rightarrow \mathcal{D}_c(\text{BGL}_d)_{\text{eff}}[U^{-1}] \rightarrow \mathcal{E}^\otimes$ preserves finite colimits. Hence we have a fully faithful functor

$$\alpha : \text{Map}_{\text{CAlg}(\widehat{\text{Cat}_\infty})}^{\text{ex}}(\mathcal{D}_c^\otimes(\text{BGL}_d)_{\text{eff}}[U^{-1}], \mathcal{E}^\otimes) \rightarrow \text{Map}_{\text{CAlg}(\widehat{\text{Cat}_\infty})}^{\text{ex}}(\mathcal{D}_c^\otimes(\text{BGL}_d)_{\text{eff}}, \mathcal{E}^\otimes)$$

where by “ex” indicates the full subcategory spanned by exact functors, i.e., functors which preserve finite colimits. The essential image consists of those functors $F : \mathcal{D}_c^\otimes(\text{BGL}_d)_{\text{eff}} \rightarrow \mathcal{E}^\otimes$ such that $F(U)$ is invertible. Since $\mathcal{D}(\text{BGL}_d)$ is compactly generated, the symmetric

monoidal Ind-category $\text{Ind}(\mathcal{D}_c^\otimes(BGL_d))$ (cf. [30, 5.3.6.8], [31, 4.8.1.13, 4.8.1.14]) is equivalent to $\mathcal{D}^\otimes(BGL_d)$. Similarly, $\text{Ind}(\mathcal{D}_c^\otimes(BGL_d)_{\text{eff}})$ is equivalent to $\mathcal{D}^\otimes(BGL_d)_{\text{eff}}$. The left Kan extension $\text{Ind}(\mathcal{D}_c^\otimes(BGL_d)_{\text{eff}}) \simeq \mathcal{D}^\otimes(BGL_d)_{\text{eff}} \rightarrow \mathcal{E}^\otimes$ (cf. [31, 4.8.1.13]) preserves small colimits if and only if the composite $\mathcal{D}_c^\otimes(BGL_d)_{\text{eff}} \rightarrow \mathcal{E}^\otimes$ preserves finite colimits (see [31, the proof of 1.1.3.6]). Thus we have a homotopy equivalence

$$\beta : \text{Map}_{\text{CAlg}(\text{Pr}^L)}(\mathcal{D}^\otimes(BGL_d)_{\text{eff}}, \mathcal{E}^\otimes) \simeq \text{Map}_{\text{CAlg}(\widehat{\text{Cat}_\infty})}^{\text{ex}}(\mathcal{D}_c^\otimes(BGL_d)_{\text{eff}}, \mathcal{E}^\otimes)$$

for any $\mathcal{E}^\otimes \in \text{CAlg}(\text{Pr}_S^L)$. Similarly, we have

$$\gamma : \text{Map}_{\text{CAlg}(\text{Pr}^L)}(\mathcal{D}^\otimes(BGL_d), \mathcal{E}^\otimes) \simeq \text{Map}_{\text{CAlg}(\widehat{\text{Cat}_\infty})}^{\text{ex}}(\mathcal{D}_c^\otimes(BGL_d), \mathcal{E}^\otimes).$$

Combining these α, β and γ , we obtain a homotopy equivalence

$$\text{Map}_{\text{CAlg}(\text{Pr}^L)}(\mathcal{D}^\otimes(BGL_d), \mathcal{E}^\otimes) \rightarrow \text{Map}_{\text{CAlg}(\text{Pr}^L)}^U(\mathcal{D}^\otimes(BGL_d)_{\text{eff}}, \mathcal{E}^\otimes)$$

induced by $\mathcal{D}^\otimes(BGL_d)_{\text{eff}} \rightarrow \mathcal{D}^\otimes(BGL_d)$. Note that by [35, 4.25] $\mathcal{D}^\otimes(BGL_d)_{\text{eff}}[U^{-1}]$ is stable. Hence $\mathcal{D}^\otimes(BGL_d)_{\text{eff}}[U^{-1}] \simeq \mathcal{D}^\otimes(BGL_d)$. \square

Proposition 3.16. *We adopt notation as above. In particular, $T = \{V_\lambda[r]\}_{\lambda, r \in \mathbb{Z}}$ where λ run over Young diagrams having more than d rows. Let L be the d -fold wedge product of I^1 in $\mathcal{D}(B\Sigma, k)$. Then there exists natural equivalence*

$$\mathcal{D}^\otimes(B\Sigma, k)_T[L^{-1}] \simeq \mathcal{D}^\otimes(BGL_d).$$

Proof. Combine Proposition 3.14 and 3.15. \square

Remark 3.17. If we identify $\mathcal{D}^\otimes(B\Sigma, k)$ with $\text{Fun}(\text{Free}(\Delta^0)^{op}, \mathcal{S}) \otimes \text{Mod}_k^\otimes$ by Proposition 3.10, we have $(\text{Fun}(\text{Free}(\Delta^0)^{op}, \mathcal{S}) \otimes \text{Mod}_k^\otimes)_T[L^{-1}] \simeq \mathcal{D}^\otimes(BGL_d)$.

Proof of Theorem 3.1. Consider the sequence of functors

$$\Delta^0 \rightarrow \text{Free}(\Delta^0) \rightarrow \text{Fun}(\text{Free}(\Delta^0)^{op}, \text{Mod}_k) \simeq \mathcal{D}(B\Sigma, k) \xrightarrow{s} \mathcal{D}(B\Sigma, k)_T \xrightarrow{t} \mathcal{D}(B\Sigma, k)_T[L^{-1}].$$

The left functor is induced by the adjoint pair $\text{Free} : \text{Cat}_\infty \rightleftarrows \text{CAlg}(\text{Cat}_\infty) : \text{forget}$, $\text{Free}(\Delta^0) \rightarrow \text{Fun}(\text{Free}(\Delta^0)^{op}, \text{Mod}_k)$ is the “natural” functor, and the middle equivalence follows from Proposition 3.8 and 3.10. The functors s and t are left adjoint functors arising from the localization and the inversion respectively. The composition with this sequence gives rise to

$$\alpha : \text{Map}_{\text{CAlg}(\text{Pr}_k^L)}^\otimes(\mathcal{D}(B\Sigma, k)_T[L^{-1}], \mathcal{C}^\otimes) \rightarrow \text{Map}(\Delta^0, \mathcal{C}) = \mathcal{C}^\simeq$$

for any $\mathcal{C}^\otimes \in \text{CAlg}(\text{Pr}_k^L)$. Combining Proposition 3.7, Lemma 3.12, and universal properties, we deduce that α is fully faithful and its essential image is $\mathcal{C}_{\wedge, d}^\simeq$. Note that through the equivalence $\mathcal{D}(B\Sigma, k)_T \simeq \mathcal{D}(BGL_d)_{\text{eff}}$, I^1 corresponds to K , and thus L corresponds to U . Finally, according to Proposition 3.16, $\mathcal{D}^\otimes(B\Sigma, k)_T[L^{-1}] \simeq \mathcal{D}^\otimes(BGL_d) = \text{Rep}^\otimes(\text{GL}_d)$. Therefore, our assertion follows. \square

4. TANNAKIAN CHARACTERIZATION

4.1. In this Section we prove Theorem 1.4; see Theorem 4.5. We also describe an explicit presentation (construction) of A and G in $[\text{Spec } A/G]$ in Theorem 4.1 and 4.5. We begin by treating its algebraic version, that is, the case when a fine ∞ -category admits a single wedge-finite generator:

Theorem 4.1. *Let \mathcal{C}^\otimes be a k -linear symmetric monoidal stable presentable ∞ -category. That is, \mathcal{C}^\otimes belongs to $\text{CAlg}(\text{Pr}_k^L)$. Then the following conditions are equivalent:*

- (1) *There exists a wedge-finite object C such that \mathcal{C}^\otimes is generated by $\{C, C^\vee\}$ as a symmetric monoidal stable presentable ∞ -category. A unit object $1_{\mathcal{C}}$ is a compact object.*

- (2) *There exist a stack $[\mathrm{Spec} A/G]$ where a reductive algebraic group G over k acts on $\mathrm{Spec} A$ with $A \in \mathrm{CAlg}_k$ and an equivalence $\mathcal{C}^\otimes \simeq \mathrm{QC}^\otimes([\mathrm{Spec} A/G])$ in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$.*
- (3) *There exist a stack $[\mathrm{Spec} A/\mathrm{GL}_d]$ where the general linear group GL_d for some $d \geq 0$ acts on $\mathrm{Spec} A$ with $A \in \mathrm{CAlg}_k$ and an equivalence $\mathcal{C}^\otimes \simeq \mathrm{QC}^\otimes([\mathrm{Spec} A/\mathrm{GL}_d])$ in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$.*

Remark 4.2. The conditions in Theorem 4.1 are equivalent to one more condition, see Corollary 6.6.

Proof. The implication from (3) to (2) is obvious.

We will prove that (2) implies (1). Let V be a finite dimensional faithful representation of G . If we think of V and V^\vee as objects in $\mathrm{QC}(BG)$, then $\mathrm{QC}^\otimes(BG)$ is generated by V and V^\vee as a symmetric monoidal stable presentable ∞ -category. Let $\mathrm{QC}^\otimes(BG) \rightarrow \mathrm{QC}^\otimes([\mathrm{Spec} A/G]) = \mathrm{Mod}_A^\otimes(\mathrm{QC}(BG))$ be the symmetric monoidal functor (informally) given by $M \mapsto A \otimes M$. Since V is wedge-finite in $\mathrm{QC}^\otimes(BG)$, $A \otimes V$ is wedge-finite. Observe that $\mathrm{QC}^\otimes([\mathrm{Spec} A/G])$ is generated by $A \otimes V$ and $A \otimes V^\vee$ as a symmetric monoidal stable presentable ∞ -category. For the present, we assume that $A \otimes V^{\otimes n}$ and $A \otimes (V^\vee)^{\otimes n}$ are compact. We will prove that for any $N \in \mathrm{QC}([\mathrm{Spec} A/G])$, the condition

$$\mathrm{Hom}_{\mathrm{h}(\mathrm{QC}([\mathrm{Spec} A/G])}(A \otimes V^{\otimes n}, N[r]) = 0 \quad \text{and} \quad \mathrm{Hom}_{\mathrm{h}(\mathrm{QC}([\mathrm{Spec} A/G])}(A \otimes (V^\vee)^{\otimes n}, N[r]) = 0$$

for any $n \geq 0$ and any $r \in \mathbb{Z}$ implies $N \simeq 0$ (cf. Remark 1.9). Consider the adjoint pair $A \otimes (-) : \mathrm{QC}(BG) \rightleftarrows \mathrm{QC}^\otimes([\mathrm{Spec} A/G]) : U$ where U is the forgetful functor. The vanishing

$$\mathrm{Hom}_{\mathrm{h}(\mathrm{QC}(BG))}(V^{\otimes n}, U(N[r])) = 0 \quad \text{and} \quad \mathrm{Hom}_{\mathrm{h}(\mathrm{QC}(BG))}((V^\vee)^{\otimes n}, U(N[r])) = 0$$

for any $n \geq 0$ and $r \in \mathbb{Z}$ implies $U(N) = 0$. Using the adjoint pair we conclude that $\mathrm{QC}^\otimes([\mathrm{Spec} A/G])$ is generated by $A \otimes V^{\otimes n}$ and $A \otimes (V^\vee)^{\otimes n}$ ($n \geq 0$) as a stable presentable ∞ -category. Thus $\mathrm{QC}^\otimes([\mathrm{Spec} A/G])$ is generated by $A \otimes V$ and $A \otimes V^\vee$ as a symmetric monoidal stable presentable ∞ -category. Now we will observe that $A \otimes V^{\otimes n}$ and $A \otimes (V^\vee)^{\otimes n}$ are compact. Taking account of this adjoint pair and the fact that (i) a unit in $\mathrm{QC}(BG)$ is compact, (ii) U preserves colimits, we see that a unit in $\mathrm{QC}([\mathrm{Spec} A/G])$ is compact. Consequently, every dualizable objects is compact. It follows that $A \otimes V^{\otimes n}$ and $A \otimes (V^\vee)^{\otimes n}$ are compact. Hence (2) implies (1).

Finally, we will prove that (3) follows from (1). Suppose that there is a d -dimensional wedge-finite object C such that \mathcal{C}^\otimes is generated by C and C^\vee . By Theorem 3.1 there is a morphism $F : \mathrm{QC}^\otimes(B\mathrm{GL}_d) \rightarrow \mathcal{C}^\otimes$ in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$ which carries the standard representation of GL_d to C . It is unique up to a contractible space of choice. We apply Proposition 2.1 to F . To this end, let us verify an existence of a small set of compact and dualizable objects generating $\mathrm{QC}(B\mathrm{GL}_d)$ as a stable presentable ∞ -category; $\{V^{\otimes n}, (V^\vee)^{\otimes n}\}_{n \geq 0}$ generates $\mathrm{QC}(B\mathrm{GL}_d)$ as a stable presentable ∞ -category. Also, $F(V^{\otimes n})$ and $F((V^\vee)^{\otimes n})$ are compact (notice that the compactness of the unit implies that every dualizable object is compact). If G denotes the right adjoint of F and $1_{\mathcal{C}}$ denotes a unit of \mathcal{C} , we let $A = G(1_{\mathcal{C}})$. Then since $1_{\mathcal{C}}$ belongs to $\mathrm{CAlg}(\mathcal{C})$, G is a lax symmetric monoidal functor (by relative adjoint functor theorem [31, 7.3.2.6]) which induces $G : \mathrm{CAlg}(\mathcal{C}) \rightarrow \mathrm{CAlg}(\mathrm{QC}(B\mathrm{GL}_d))$. Therefore, A belongs to $\mathrm{CAlg}(\mathrm{QC}(B\mathrm{GL}_d))$. According to Proposition 2.1 there exists an equivalence $\mathrm{Mod}_A^\otimes(\mathrm{QC}(B\mathrm{GL}_d)) \simeq \mathcal{C}^\otimes$ in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$. Hence $\mathrm{QC}^\otimes([\mathrm{Spec} A/\mathrm{GL}_d]) \simeq \mathrm{Mod}_A^\otimes(\mathrm{QC}(B\mathrm{GL}_d)) \simeq \mathcal{C}^\otimes$ for the corresponding stack $[\mathrm{Spec} A/\mathrm{GL}_d]$. \square

Remark 4.3. By the proof, we can choose d in the condition (3) to be the dimension of a wedge-finite object C in the condition (1).

Definition 4.4. When \mathcal{C}^\otimes satisfies conditions in Theorem 4.1, we shall refer to \mathcal{C}^\otimes as a *fine algebraic ∞ -category*.

Theorem 4.5. *Let \mathcal{C}^\otimes be a k -linear symmetric monoidal stable presentable ∞ -category. That is, \mathcal{C}^\otimes belongs to $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$. The followings are equivalent to one another:*

- (1) \mathcal{C}^\otimes is a fine ∞ -category.

- (2) There exist a quotient stack $X = [\mathrm{Spec} A/G]$ where a pro-reductive group G acts on an affine derived scheme $\mathrm{Spec} A$ with $A \in \mathrm{CAlg}_k$ and an equivalence $\mathcal{C}^\otimes \simeq \mathrm{QC}^\otimes(X)$ in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$.

The result we need from [2] is the following:

Proposition 4.6 ([2]). *Let X and Y be a perfect derived stack over k . Suppose that $X = [\mathrm{Spec} A/G]$ and $Y = [\mathrm{Spec} B/H]$ where $A, B \in \mathrm{CAlg}_k$, and G and H are pro-reductive groups over k . Let $p_X^* : \mathrm{QC}^\otimes(X) \rightarrow \mathrm{QC}^\otimes(X \times_k Y)$ and $p_Y^* : \mathrm{QC}^\otimes(Y) \rightarrow \mathrm{QC}^\otimes(X \times_k Y)$ be the pullback functors of natural projections. Let $\mathrm{QC}^\otimes(X) \otimes_k \mathrm{QC}^\otimes(Y)$ denote the coproduct of $\mathrm{QC}^\otimes(X)$ and $\mathrm{QC}^\otimes(Y)$ in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$ and*

$$F : \mathrm{QC}^\otimes(X) \otimes_k \mathrm{QC}^\otimes(Y) \rightarrow \mathrm{QC}^\otimes(X \times_k Y)$$

the symmetric monoidal functor induced by p_X^ and p_Y^* . Then F is an equivalence.*

Proof. This assertion follows from [2, Theorem 1.2] and the proof; our notion of derived stacks is slightly different from that of [2], but the argument is applicable to our setting. For the reader's convenience we outline the proof (to fit our situation). We note that by [31, 3.2.4.7] the underlying ∞ -category of $\mathrm{QC}^\otimes(X) \otimes_k \mathrm{QC}^\otimes(Y)$ is a tensor product of $\mathrm{QC}(X)$ and $\mathrm{QC}(Y)$ in $\mathrm{Pr}_k^{\mathrm{L}}$. It is enough to prove the underlying functor of F is an equivalence of ∞ -categories. When $X = BG$ and $Y = BH$ where G and H are reductive algebraic groups, an equivalence of F is a special case of [2, Theorem 1.2]. Suppose that G and H are pro-reductive, and $G = \varprojlim G_\alpha$ and $H = \varprojlim H_\beta$ are filtered projective limits of reductive algebraic groups such that projections $G \rightarrow G_\alpha$ and $H \rightarrow H_\beta$ are surjective. The case of $X = BG$ and $Y = BH$ (or the general case) also follows from the proof of [2, Theorem 1.2]. We here give another (ad hoc) argument. By Lemma 4.7 below $\mathrm{QC}(BG)$ is a colimit of $\mathrm{QC}(BG_\alpha)$ in $\mathrm{Pr}_k^{\mathrm{L}}$, and $\mathrm{QC}(BH)$ is a colimit of $\mathrm{QC}(BH_\beta)$ in $\mathrm{Pr}_k^{\mathrm{L}}$. Since the tensor product \otimes_k preserves colimits separately in each variable, by using presentations as colimits we see that $\mathrm{QC}(BG) \otimes_k \mathrm{QC}(BH) \simeq \mathrm{QC}(B(G \times_k H)) \simeq \mathrm{QC}(BG \times_k BH)$. In particular, $\mathrm{QC}^\otimes(BG) \otimes_k \mathrm{QC}^\otimes(BH) \simeq \mathrm{QC}^\otimes(BG \times_k BH)$. Next we consider the general case $X = [\mathrm{Spec} A/G]$ and $Y = [\mathrm{Spec} B/H]$. Let $A \boxtimes B$ denote the tensor product of $p_{BG}^*(A)$ and $p_{BH}^*(B)$ as objects $\mathrm{CAlg}(\mathrm{QC}(BG \times_k BH))$, where p_{BG} and p_{BH} are natural projections. Then $A \boxtimes B$ in $\mathrm{CAlg}(\mathrm{QC}(BG \times_k BH))$ gives rise to the quotient stack $[\mathrm{Spec}(A \boxtimes B)/(G \times_k H)]$, that is equivalent to $[\mathrm{Spec} A/G] \times_k [\mathrm{Spec} B/H]$. Then we have a natural equivalence

$$\mathrm{QC}([\mathrm{Spec} A/G] \times_k [\mathrm{Spec} B/H]) \simeq \mathrm{Mod}_{A \boxtimes B}(\mathrm{QC}(BG \times_k BH)),$$

and by [2, Proposition 4.1 (2)] both sides are also equivalent to

$$\mathrm{Mod}_{p_{BG}^*(A)}(\mathrm{QC}(BG \times_k BH)) \otimes_{\mathrm{QC}(BG \times_k BH)} \mathrm{Mod}_{p_{BH}^*(B)}(\mathrm{QC}(BG \times_k BH)).$$

In addition, according to [2, Proposition 4.1 (1)] we have

$$\mathrm{Mod}_{p_{BG}^*(A)}(\mathrm{QC}(BG \times_k BH)) \simeq \mathrm{Mod}_A(\mathrm{QC}(BG)) \otimes_{\mathrm{QC}(BG)} \mathrm{QC}(BG \times_k BH)$$

and

$$\mathrm{Mod}_{p_{BH}^*(B)}(\mathrm{QC}(BG \times_k BH)) \simeq \mathrm{Mod}_B(\mathrm{QC}(BH)) \otimes_{\mathrm{QC}(BH)} \mathrm{QC}(BG \times_k BH).$$

Using these equivalences together with $\mathrm{QC}(BG) \otimes_k \mathrm{QC}(BH) \simeq \mathrm{QC}(BG \times_k BH)$, we obtain

$$\mathrm{QC}([\mathrm{Spec} A/G] \times_k [\mathrm{Spec} B/H]) \simeq \mathrm{Mod}_A(\mathrm{QC}(BG)) \otimes_k \mathrm{Mod}_B(\mathrm{QC}(BH))$$

where the right hand side is naturally equivalent to $\mathrm{QC}([\mathrm{Spec} A/G]) \otimes_k \mathrm{QC}([\mathrm{Spec} B/H])$. \square

Lemma 4.7. *Let $G = \varprojlim_{\beta < \alpha} G_\beta$ be a limit of pro-reductive algebraic groups indexed by a limit ordinal α . Namely, $G = G_\alpha$ is a limit of the sequence*

$$\dots \rightarrow G_{\beta+1} \rightarrow G_\beta \rightarrow \dots \rightarrow G_1 \rightarrow G_0$$

as an affine group scheme, where for any $\beta < \alpha$, G_β is a pro-reductive group over k . Suppose that for any $\gamma < \beta$ the morphism $G_\beta \rightarrow G_\gamma$ is surjective. Then the pullback functors induce an equivalence

$$\varinjlim \mathrm{Perf}^\otimes(BG_\beta) \rightarrow \mathrm{Perf}^\otimes(BG)$$

where the left hand side is the colimit in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$. Here $\mathrm{Perf}^\otimes(BG_\beta)$ denotes the stable subcategory of $\mathrm{QC}^\otimes(BG_\beta)$ spanned by dualizable objects (note that it coincides with $\mathcal{D}_c(BG_\beta)$).

Moreover, the above equivalence is extended to an equivalence

$$\varinjlim \mathrm{QC}^\otimes(BG_\beta) \rightarrow \mathrm{QC}^\otimes(BG)$$

in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$ where the left hand side is the colimit in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$.

Proof. Let $G = G_\alpha$. Note first that for $\alpha \geq \beta \geq \gamma$, the surjective map $G_\beta \rightarrow G_\gamma$ induces a fully faithful pullback functor $\mathrm{Perf}(BG_\gamma) \rightarrow \mathrm{Perf}(BG_\beta)$. In fact, taking account of the semi-simplicity of the representations of G_β , we see that any object W in $\mathrm{Perf}(BG_\beta)$ has the form $V_0[r_0] \oplus \dots \oplus V_n[r_n]$ where V_i is a finite dimensional irreducible representation of G_β and r_i is an integer for any $n \geq i \geq 0$. Moreover, $\mathrm{Hom}_{\mathrm{h}(\mathrm{QC}(BG_\beta))}(V_i, V_i[r])$ is a division algebra for $r = 0$, and it is zero if $r \neq 0$. Thus we conclude that $\mathrm{Perf}(BG_\gamma) \rightarrow \mathrm{Perf}(BG_\beta)$ is fully faithful, and its essential image is spanned by those objects which has the form $V_0[r_0] \oplus \dots \oplus V_n[r_n]$ where V_i is an irreducible representation of G_β arising from the factorization $G_\beta \rightarrow G_\gamma$, and r_i is an integer for any $n \geq i \geq 0$ (keep in mind that an exact functor between stable ∞ -categories is an equivalence if and only if the induced functor between their homotopy categories is an equivalence, see e.g. [24]). To prove an equivalence $\varinjlim \mathrm{Perf}^\otimes(BG_\beta) \rightarrow \mathrm{Perf}^\otimes(BG)$, by [31, 3.2.3.1] it is enough to show that the colimit $\varinjlim \mathrm{Perf}^\otimes(BG_\beta)$ in Cat_∞ is naturally equivalent to $\mathrm{Perf}(BG)$. For this, since each $\mathrm{Perf}(BG_\gamma) \rightarrow \mathrm{Perf}(BG_\beta)$ is fully faithful, it will suffice to observe that every object C in $\mathrm{Perf}(BG_\alpha)$ belongs to $\mathrm{Perf}(BG_\beta)$ for some $\beta < \alpha$. Let A_β denote the ring of functions on G_β , that is endowed with a structure of a commutative Hopf algebra. The formulation $G_\alpha = \varprojlim_{\beta < \alpha} G_\beta$ of the limit gives rise to $A_\alpha = \cup_{\beta < \alpha} A_\beta$, where we regard A_β as a Hopf subalgebra of A_α . Let $W \simeq V_0[r_0] \oplus \dots \oplus V_n[r_n]$ be an object in $\mathrm{Perf}(BG_\alpha)$ where V_i is a finite dimensional irreducible representation of G_α , and r_i is an integer for any $n \geq i \geq 0$. Each V_i is finite dimensional and thus the corresponding coaction $V_i \rightarrow V_i \otimes A_\alpha$ factors through $V_i \rightarrow V_i \otimes H_i$ for a finitely generated commutative Hopf algebra $H_i \subset A_\alpha$. Let $\{x_1^i, \dots, x_{s_i}^i\}$ be the set of generators of H_i as a commutative k -algebra. If we choose a sufficiently large $\beta < \alpha$, x_j^i lies in A_β for any i and j . Therefore all H_i are contained in A_β . It follows that W belongs to $\mathrm{Perf}(BG_\beta)$.

Next we prove that $\varinjlim_{\beta < \alpha} \mathrm{QC}^\otimes(BG_\beta) \rightarrow \mathrm{QC}^\otimes(BG)$. By taking a left Kan extension [31, 4.8.1.13] $\mathrm{Perf}^\otimes(BG_\beta) \rightarrow \mathrm{QC}^\otimes(BG)$ is extended to $\mathrm{Ind}(\mathrm{Perf}^\otimes(BG_\beta)) \rightarrow \mathrm{QC}^\otimes(BG)$ which preserves small colimits. Observe that $\mathrm{Ind}(\mathrm{Perf}^\otimes(BG_\beta)) \simeq \mathrm{QC}^\otimes(BG_\beta)$. Since objects in $\mathrm{Perf}(BG_\beta)$ is compact in $\mathrm{QC}(BG_\beta)$, by [30, 5.3.4.12] the left Kan extension $\mathrm{Ind}(\mathrm{Perf}(BG_\beta)) \rightarrow \mathrm{QC}(BG_\beta)$ is fully faithful. Note that G_β is a pro-reductive group, and therefore the abelian category of representations of G_β is semi-simple. As is well-known, every representation W of G_β can be described as a filtered colimit $\varinjlim V_z$ of finite dimensional subrepresentations V_z . Thus $\mathrm{Ind}(\mathrm{Perf}(BG_\beta)) \rightarrow \mathrm{QC}(BG_\beta)$ is essentially surjective. Apply the equivalence

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Cat}_\infty)}^{\mathrm{ex}}(\mathrm{Perf}^\otimes(BG_\beta), \mathcal{D}^\otimes) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})}(\mathrm{QC}^\otimes(BG_\beta), \mathcal{D}^\otimes)$$

for $\mathcal{D}^\otimes \in \mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$ and $\alpha \geq \beta$ to deduce that $\mathrm{QC}^\otimes(BG)$ is a filtered colimit $\varinjlim_{\beta < \alpha} \mathrm{QC}^\otimes(BG_\beta)$ in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$. Here the superscript “ex” indicates the full subcategory spanned by exact functors. By [31, 4.2.3.5, 3.2.3.1] $\mathrm{QC}^\otimes(BG_\beta)$ is a colimit $\varinjlim_{\beta < \alpha} \mathrm{QC}^\otimes(BG_\beta)$ in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$. \square

Proof of Theorem 4.5. We prove that (1) implies (2). Let \mathcal{C}^\otimes be an object in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$. Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a small set of wedge-finite objects such that \mathcal{C}^\otimes is generated by $\{C_\lambda, C_\lambda^\vee\}_{\lambda \in \Lambda}$. Choose a bijective map $\Lambda \simeq \alpha$ where α is a cardinal. We replace $\{C_\lambda\}_{\lambda \in \Lambda}$ by $\{C_\beta\}_{\beta < \alpha}$. We

will construct a pro-reductive group G and a morphism $F : \text{Rep}^\otimes(G) \simeq \text{QC}^\otimes(BG) \rightarrow \mathcal{C}^\otimes$ by transfinite induction.

Let n_β be the dimension of the wedge-finite object C_β . Invoking Theorem 3.1, C_0 gives rise to a morphism $F_1 : \text{QC}^\otimes(B\text{GL}_{n_0}) \rightarrow \mathcal{C}^\otimes$ in $\text{CAlg}(\text{Pr}_k^{\text{L}})$ which carries the standard representation of GL_{n_0} (placed in degree zero) to C_0 . Set $G_1 := \text{GL}_{n_0}$.

Suppose that G_β and $F_\beta : \text{QC}^\otimes(BG_\beta) \rightarrow \mathcal{C}^\otimes$ has been constructed for β . In addition, assume that $G_\beta = \varprojlim_{\gamma < \beta} G_\gamma$ if β is a limit ordinal, and $G_\beta = G_{\beta-1} \times_k \text{GL}_{n_{\beta-1}}$ if otherwise (by convention G_0 is trivial). Moreover, suppose that $G_\beta \rightarrow G_\gamma$ is surjective for $\gamma < \beta$. By Theorem 3.1 we have $F'_{\beta+1} : \text{QC}^\otimes(B\text{GL}_{n_\beta}) \rightarrow \mathcal{C}^\otimes$ which carries the standard representation of GL_{n_β} to C_β . Using Proposition 4.6 we prove that

$$\text{QC}^\otimes(BG_\beta \times_k B\text{GL}_{n_\beta}) \simeq \text{QC}^\otimes(BG_\beta) \otimes_k \text{QC}^\otimes(B\text{GL}_{n_\beta}).$$

Then the “coproduct” of F_β and $F'_{\beta+1}$ induces

$$F_{\beta+1} : \text{QC}^\otimes(BG_\beta \times_k B\text{GL}_{n_\beta}) \simeq \text{QC}^\otimes(BG_\beta) \otimes_k \text{QC}^\otimes(B\text{GL}_{n_\beta}) \rightarrow \mathcal{C}^\otimes.$$

Note that by [30, 5.5.8.11, 5.5.8.12] $B(G_\beta \times_k \text{GL}_{n_\beta}) \simeq BG_\beta \times_k B\text{GL}_{n_\beta}$. We define $G_{\beta+1}$ to be $G_\beta \times_k \text{GL}_{n_\beta}$. If $p_\beta : G_{\beta+1} = G_\beta \times_k \text{GL}_{n_\beta} \rightarrow G_\beta$ is the first projection, then we have a commutative diagram (i.e. 2-cell)

$$\begin{array}{ccc} \text{QC}^\otimes(BG_\beta) & \xrightarrow{p_\beta^*} & \text{QC}^\otimes(BG_{\beta+1}) \\ & \searrow F_\beta & \downarrow F_{\beta+1} \\ & & \mathcal{C}^\otimes \end{array}$$

in $\text{CAlg}(\text{Pr}_k^{\text{L}})$.

Let β be a limit ordinal. Suppose that a linearly ordered sequence indexed by β

$$\cdots \rightarrow G_{\gamma+1} \xrightarrow{p_\gamma} G_\gamma \rightarrow \cdots \xrightarrow{p_1} G_1$$

of pro-reductive groups and

$$\begin{array}{ccccccc} \text{QC}^\otimes(BG_1) & \xrightarrow{p_1^*} & \cdots & \longrightarrow & \text{QC}^\otimes(BG_\gamma) & \xrightarrow{p_\gamma^*} & \text{QC}^\otimes(BG_{\gamma+1}) \longrightarrow \cdots \\ & \searrow F_0 & & \swarrow F_\gamma & & \swarrow F_{\gamma+1} & \\ & & & \mathcal{C}^\otimes & & & \end{array}$$

in $\text{CAlg}(\text{Pr}_k^{\text{L}})_{/\mathcal{C}^\otimes}$ have been defined. Suppose that each p_γ is surjective. Let $G_\beta := \varprojlim_{\gamma < \beta} G_\gamma$. Then by Lemma 4.7 $\varinjlim_{\gamma < \beta} \text{QC}^\otimes(BG_\gamma) \simeq \text{QC}^\otimes(BG_\beta)$. Hence by the universal property of the colimit and Lemma 4.7 the above diagram induces a morphism $\text{QC}^\otimes(BG_\beta) \rightarrow \mathcal{C}^\otimes$ in $\text{CAlg}(\text{Pr}_k^{\text{L}})$. By transfinite induction we have a pro-reductive group $G := G_\alpha$ and $F := F_\alpha : \text{QC}^\otimes(BG) \rightarrow \mathcal{C}^\otimes$.

Next we prove that $F : \text{QC}^\otimes(BG) \rightarrow \mathcal{C}^\otimes$ satisfies

- there is a small set of compact and dualizable objects $\{I_\lambda\}_{\lambda \in \Lambda}$ which generates $\text{QC}(BG)$ as a stable presentable ∞ -category,
- $\{F(I_\lambda)\}_{\lambda \in \Lambda}$ is a set of compact objects in \mathcal{C} which generates \mathcal{C} as a stable presentable ∞ -category.

If we define $\{I_\lambda\}_{\lambda \in \Lambda}$ to be the set of (finite dimensional) irreducible representations of G , then the first condition is satisfied. To check the second condition, note that there are natural surjective homomorphisms $G \rightarrow G_{\beta+1} = G_\beta \times_k \text{GL}_{n_\beta} \rightarrow \text{GL}_{n_\beta}$. The pullback of the composite induces an irreducible representation of G from the standard representation of GL_{n_β} . Thus $\{C_\lambda, C_\lambda^\vee\}_{\lambda \in \Lambda}$ is contained in the essential image of F . Hence the second condition is satisfied (notice that dualizable objects are compact in \mathcal{C}). Let H be a right adjoint functor of F . As in

the proof of Theorem 4.1, $H(1_C)$ belongs to $\mathrm{CAlg}(\mathrm{QC}(BG)) \simeq \mathrm{CAlg}(\mathrm{Rep}(G))$. Now we apply to Proposition 2.1 to F and obtain an equivalence $\mathrm{QC}^\otimes([\mathrm{Spec} A/G]) \simeq \mathrm{Mod}_A^\otimes(\mathrm{Rep}(G)) \simeq \mathcal{C}^\otimes$ where $[\mathrm{Spec} A/G]$ is a stack corresponding to $A \in \mathrm{CAlg}(\mathrm{Rep}(G))$.

Next we prove that (2) implies (1). As in the proof of Theorem 4.1, if $\{I_\lambda\}_{\lambda \in \Lambda}$ is the set of irreducible representations of G , then $\{A \otimes I_\lambda\}_{\lambda \in \Lambda}$ is the set of compact and dualizable objects which generates $\mathrm{Mod}_A(\mathrm{QC}(BG)) = \mathrm{QC}([\mathrm{Spec} A/G])$ as a stable presentable ∞ -category. Every $A \otimes I_\lambda$ is wedge-finite. Finally, the unit of $\mathrm{QC}([\mathrm{Spec} A/G])$ is compact since the unit in $\mathrm{QC}(BG)$ is compact (use adjoint pair $\mathrm{QC}(BG) \rightleftarrows \mathrm{QC}([\mathrm{Spec} A/G])$). \square

4.2. For a fine ∞ -category \mathcal{C}^\otimes there are many choices of quotient forms $[\mathrm{Spec} A/G]$ such that $\mathcal{C}^\otimes \simeq \mathrm{QC}^\otimes([\mathrm{Spec} A/G])$. One pleasant feature of our construction in the proof of Theorem 4.1 and Theorem 4.5 is that given a set of wedge-finite generators we have an explicit quotient form $[\mathrm{Spec} A/G]$. For example, as in the proof of Theorem 4.1 and Theorem 4.5, we can take G to be a product of general linear groups. It is useful for many applications. We will describe A in terms of a given set of generators.

To begin, we consider the case when a fine ∞ -category \mathcal{C}^\otimes has a single wedge-finite (compact) generator C ; the fine algebraic case. Let d be the dimension of C .

Let λ be the Young diagram with n boxes. As in the case of Alt^n , we let $\mathbb{S}_\lambda C$ be the image of the associated idempotent map $C^{\otimes n} \rightarrow C^{\otimes n}$ (in the idempotent complete homotopy category of \mathcal{C}^\otimes). To a Young diagram λ with n boxes, by choosing the lift to a Young tableau we associate the Young symmetrizer $c_\lambda \in \mathbb{Q}[\Sigma_n]$ which satisfies $c_\lambda c_\lambda = a_\lambda c_\lambda$ where a_λ is a certain rational number (cf. [12, Lecture 4]). This $a_\lambda^{-1} c_\lambda$ gives an idempotent map $C^{\otimes n} \rightarrow C^{\otimes n}$ via permutation. We define $\mathbb{S}_\lambda C$ to be $\mathrm{Ker}(1 - a_\lambda^{-1} c_\lambda)$.

Let $\mathrm{Hom}_\mathcal{C}(-, -)$ denote the hom complex which belongs to Mod_k . Namely, for any $D \in \mathcal{C}$, we have the adjoint pair

$$D \otimes s(-) : \mathrm{Mod}_k \rightleftarrows \mathcal{C} : \mathrm{Hom}_\mathcal{C}(D, -)$$

where s is the “structure” functor $\mathrm{Mod}_k^\otimes \rightarrow \mathcal{C}^\otimes$, and the existence of the right adjoint functor $\mathrm{Hom}_\mathcal{C}(D, -)$ is implied by the adjoint functor theorem and the fact that $D \otimes s(-)$ preserves small colimits. By the highest weight theory, the set of isomorphism classes of irreducible representations of GL_d bijectively corresponds to the set

$$\mathbb{Z}_*^{\oplus d} := \{\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^{\oplus d} \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d\}.$$

That is, when $\lambda_d \geq 0$, $\lambda = (\lambda_1, \dots, \lambda_d)$ determines a partition of $\lambda_1 + \dots + \lambda_d$, and it corresponds to the irreducible representation $\mathbb{S}_\lambda K$ where K is the standard representation of GL_d . When $\lambda_d < 0$, $\lambda^+ = (\lambda_1 - \lambda_d, \lambda_2 - \lambda_d, \dots, \lambda_d - \lambda_d)$ determined a partition of $(\lambda_1 + \dots + \lambda_d) - d\lambda_d$ (regarded as a Young diagram), and λ corresponds to the irreducible representation $(\mathbb{S}_{\lambda^+} K) \otimes (\wedge^d K^\vee)^{\otimes (-\lambda_d)}$. If $\lambda_d < 0$, we define $\mathbb{S}_\lambda K$ to be $(\mathbb{S}_{\lambda^+} K) \otimes (\wedge^d K^\vee)^{\otimes (-\lambda_d)}$. Replacing K by a wedge-finite object C of a fine ∞ -category we define $\mathbb{S}_\lambda C$ for any $\lambda \in \mathbb{Z}_*^{\oplus d}$ in a similar way.

Proposition 4.8. *Let \mathcal{C}^\otimes be an fine algebraic ∞ -category; suppose that a fine ∞ -category \mathcal{C}^\otimes admits a single d -dimensional wedge-finite object C such that $\{C, C^\vee\}$ generates \mathcal{C}^\otimes as a symmetric monoidal stable presentable ∞ -category. Then in (3) in Theorem 4.1 we can choose a derived stack $[\mathrm{Spec} A/\mathrm{GL}_d]$ such that*

$$A \simeq \bigoplus_{\lambda \in \mathbb{Z}_*^{\oplus d}} \mathrm{Hom}_\mathcal{C}(\mathbb{S}_\lambda C, 1_C) \otimes \mathbb{S}_\lambda K$$

in $\mathrm{Rep}(\mathrm{GL}_d)$. The action of GL_d on the right hand side is given by $\mathbb{S}_\lambda K$.

Proof. In the proof of (3) \Rightarrow (1) in Theorem 4.1, using Theorem 3.1 we constructed a k -linear symmetric monoidal colimit-preserving functor $F : \mathrm{Rep}^\otimes(\mathrm{GL}_d) \rightarrow \mathcal{C}^\otimes$ sending the standard representation K to C , which has a (lax symmetric monoidal) right adjoint $G : \mathcal{C}^\otimes \rightarrow$

$\text{Rep}^\otimes(\text{GL}_d)$. Put $A = G(1_C)$. We have proved that $\text{QC}^\otimes([\text{Spec } A/\text{GL}_d]) \simeq \mathcal{C}^\otimes$ (in Theorem 4.1). To prove this Proposition, note that there exist natural equivalences

$$\begin{aligned} \text{Map}_{\text{Rep}(\text{GL}_d)}(\mathbb{S}_\alpha K, \oplus_{\lambda \in \mathbb{Z}_*^{\oplus d}} \text{Hom}_{\mathcal{C}}(\mathbb{S}_\lambda C, 1_C) \otimes \mathbb{S}_\lambda K) &\simeq \oplus_{\lambda \in \mathbb{Z}_*^{\oplus d}} \text{Map}(\mathbb{S}_\alpha K, \text{Hom}_{\mathcal{C}}(\mathbb{S}_\lambda C, 1_C) \otimes \mathbb{S}_\lambda K) \\ &\simeq \Omega^\infty \text{Hom}_{\mathcal{C}}(\mathbb{S}_\alpha C, 1_C) \\ &\simeq \text{Map}_{\mathcal{C}}(\mathbb{S}_\alpha C, 1_C) \\ &\simeq \text{Map}_{\text{Rep}(\text{GL}_d)}(\mathbb{S}_\alpha K, A) \end{aligned}$$

where the first equivalence follows from the compactness of $\mathbb{S}_\alpha K$, and the final equivalence is implied by the adjoint pair (notice also that $F(\mathbb{S}_\alpha K) = \mathbb{S}_\alpha C$). Every object $M \in \text{Rep}(\text{GL}_d)$ is a coproduct of objects $\mathbb{S}_\alpha K[r]$ with $\alpha \in \mathbb{Z}_*^{\oplus d}$ and $r \in \mathbb{Z}$. Consequently, we see that $A \simeq \oplus_{\lambda \in \mathbb{Z}_*^{\oplus d}} \text{Hom}_{\mathcal{C}}(\mathbb{S}_\lambda C, 1_C) \otimes \mathbb{S}_\lambda K$. \square

Next we treat an arbitrary fine ∞ -category. We first collect some points from the proof of Theorem 4.5: Suppose that \mathcal{C}^\otimes is a k -linear fine ∞ -category and $\{C_\lambda\}_{\lambda \in \Lambda}$ is a set of wedge-finite objects such that $\{C_\lambda, C_\lambda^\vee\}_{\lambda \in \Lambda}$ generates \mathcal{C}^\otimes as a symmetric monoidal stable presentable ∞ -category. Then we have constructed a pro-reductive group G and an adjoint pair

$$F : \text{QC}^\otimes(BG) \rightleftarrows \mathcal{C}^\otimes : H$$

where F is a k -linear symmetric monoidal (left adjoint) colimit-preserving functor. We put $A = H(1_C)$ and proved $\mathcal{C}^\otimes \simeq \text{QC}^\otimes([\text{Spec } A/G])$. By the construction, G is a product $\prod_{\lambda \in \Lambda} \text{GL}_{n_\lambda}$ where n_λ is the dimension of C_λ . Hence G has the form $\varprojlim_{S \in P_{\text{fin}}(\Lambda)} G_S$, where $P_{\text{fin}}(\Lambda)$ is the set of finite subsets of Λ , and G_S denotes the product of $\prod_{s \in S} \text{GL}_{n_s}$. The commutative Hopf algebra $\Gamma(G)$ of G is a union of Hopf subalgebras of G_S with $S \in P_{\text{fin}}(\Lambda)$. Hence every finite dimensional representation of G factors through some quotient $G \rightarrow G_S$.

Lemma 4.9. *Every irreducible representation of $G_S = \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_r}$ is of the form $p_1^*(V_1) \otimes \cdots \otimes p_r^*(V_r)$ such that V_i is an irreducible representation of GL_{n_i} and p_i is the natural projection $BG_S \rightarrow B\text{GL}_{n_i}$. The endomorphism algebra $\text{End}(p_1^*(V_1) \otimes \cdots \otimes p_r^*(V_r))$ is k .*

Remark 4.10. Consequently, every irreducible representation of $\prod_{\lambda \in \Lambda} \text{GL}_{n_\lambda}$ has the form $\otimes_{s \in S} p_s^*(V_s)$ where S is a finite set of Λ , p_s is the natural projection $B \prod_{\lambda \in \Lambda} \text{GL}_{n_\lambda} \rightarrow B\text{GL}_{n_s}$, and V_s is an irreducible representation of GL_{n_s} .

Remark 4.11. We remark also that if each V_i is an irreducible representation of GL_{n_i} , then $V_1 \otimes \cdots \otimes V_r$ is an irreducible representation of G_S . Indeed, by [2, Proposition 4.6]

$$\text{End}(p_1^*(V_1) \otimes \cdots \otimes p_r^*(V_r)) \simeq \text{End}(V_1) \otimes_k \cdots \otimes_k \text{End}(V_r) \simeq k \otimes_k \cdots \otimes_k k \simeq k.$$

Proof of Lemma 4.9. It is a standard fact, but we outline the proof for the reader's convenience. According to [2, Proposition 3.24] (and its proof) the set of objects $\{p_1^*(V_1) \otimes \cdots \otimes p_r^*(V_r)\}$ where each V_i run through irreducible representations of GL_{n_i} is a set of compact objects in $\text{QC}(BG_S)$ which generates $\text{QC}(BG_S)$ as a stable presentable ∞ -category. Thus every irreducible representation V of G_S (regarded as an object in $\text{QC}(BG_S)$) is a filtered colimits of objects in $\{p_1^*(V_1) \otimes \cdots \otimes p_r^*(V_r)[n]\}_{n \in \mathbb{Z}}$. The formulation of cohomology groups is compatible with filtered colimits, thus V is a filtered colimit of objects in $\{p_1^*(V_1) \otimes \cdots \otimes p_r^*(V_r)\}$ in the abelian category of representations of G_S . Consequently, (by semi-simplicity and irreducibility of V) we deduce that V is isomorphic to an object of the form $p_1^*(V_1) \otimes \cdots \otimes p_r^*(V_r)$. Remark 4.11 implies the second assertion. \square

Using Lemma 4.9, Remark 4.10, 4.11 we deduce the following explicit formula as in Proposition 4.8:

Proposition 4.12. *Let*

$$A_S = \bigoplus_{(\alpha_\xi) \in \prod_{\xi \in S} \mathbb{Z}_*^{\oplus n_\xi}} \mathrm{Hom}_{\mathcal{C}}(\bigotimes_{\xi \in S} \mathbb{S}_{\alpha_\xi} C_\xi, 1_{\mathcal{C}}) \otimes (\bigotimes_{\xi \in S} \mathbb{S}_{\alpha_\xi} K_\xi).$$

Here K_ξ is the standard representation of GL_{n_ξ} which we naturally regard as an irreducible representation of G . The set $\mathbb{Z}_*^{\oplus n_\xi}$ parametrizes the isomorphism classes of irreducible representations of GL_{n_ξ} . Then there exists an equivalence

$$A \simeq \varinjlim_{S \in P_{\mathrm{fin}}(\Lambda)} A_S$$

in $\mathrm{Rep}(G)$. We regard $P_{\mathrm{fin}}(\Lambda)$ as a poset by inclusions, and $S \hookrightarrow S'$ induces $A_S \rightarrow A_{S'}$ in the obvious way.

4.3. As an immediate application, we conclude this section by explaining how to construct the Tannaka dual:

Remark 4.13 (Tannaka dual). Let \mathcal{C}^\otimes be a fine ∞ -category and $F : \mathcal{C}^\otimes \rightarrow \mathrm{Mod}_k^\otimes$ a morphism in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$. Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a set of wedge-finite generator and suppose that $F(C_\lambda)$ in Mod_k is concentrated in degree zero for each $\lambda \in \Lambda$. Then the stack $[\mathrm{Spec} A/G]$ and the equivalence $\mathrm{QC}^\otimes([\mathrm{Spec} A/G]) \simeq \mathcal{C}^\otimes$ associated to \mathcal{C}^\otimes and $\{C_\lambda\}_{\lambda \in \Lambda}$ give rise to the composite $\mathrm{QC}^\otimes(BG) \rightarrow \mathrm{QC}^\otimes([\mathrm{Spec} A/G]) \simeq \mathcal{C}^\otimes \xrightarrow{F} \mathrm{Mod}_k^\otimes$ where the first functor is the pullback functor induced by the natural morphism $[\mathrm{Spec} A/G] \rightarrow BG$. By our construction this composite is equivalent to the forgetful functor. Then the right adjoint of this forgetful functor carries the unit of Mod_k to the ring of functions $\Gamma(G)$ (placed in degree zero). It yields a morphism $p : \mathrm{Spec} k \simeq [\mathrm{Spec} \Gamma(G)/G] \rightarrow [\mathrm{Spec} A/G]$. The based loop stack $\Omega_*[\mathrm{Spec} A/G] := \mathrm{Spec} k \times_{[\mathrm{Spec} A/G]} \mathrm{Spec} k$ is a derived affine group scheme (cf. [24, Appendix]), i.e., a group object in Aff_k . This construction is a G -equivariant version of bar construction. Note also that we have a natural identification $F \simeq p^*$. By the main result of [25, Theorem 4.8], this derived affine group scheme $\Omega_*[\mathrm{Spec} A/G]$ represents the automorphism group $\mathrm{Aut}(F)$ of F (see [25] for the precise formulation). We define the *Tannaka dual* of \mathcal{C}^\otimes with respect to F to be $\Omega_*[\mathrm{Spec} A/G]$.

5. SYMMETRIC MONOIDAL FUNCTORS AND CORRESPONDENCES

As observed in Introduction, a symmetric monoidal functor $\mathrm{QC}^\otimes(Y) \rightarrow \mathrm{QC}^\otimes(X)$ is not necessarily the pullback functor of a morphism $X \rightarrow Y$. For example, by Theorem 3.1 giving a k -linear symmetric monoidal functor

$$\mathrm{QC}^\otimes(B\mathrm{GL}_d) \rightarrow \mathrm{QC}^\otimes(\mathrm{Spec} k)$$

amounts to giving a d -dimensional wedge-finite object in $\mathrm{QC}^\otimes(\mathrm{Spec} k)$. Let $V[2n]$ be a d -dimensional k -vector space placed in (homological) degree $2n$. Then $V[2n]$ is a d -dimensional wedge-finite object, and it gives rise to a symmetric monoidal functor $\phi_{2n} : \mathrm{QC}^\otimes(B\mathrm{GL}_d) \rightarrow \mathrm{QC}^\otimes(\mathrm{Spec} k)$ which carries the standard representation of GL_d to $V[2n]$. On the other hand, a morphism $\mathrm{Spec} k \rightarrow B\mathrm{GL}_d$ of stacks corresponds to GL_d -torsor over $\mathrm{Spec} k$, that is, the trivial torsor. In particular, the pullback functor of $\mathrm{Spec} k \rightarrow B\mathrm{GL}_d$ sends the standard representation of GL_d to a k -vector space placed in degree zero. If $n \neq 0$, then ϕ_{2n} is not the pullback functor. This means that morphisms of stacks are not enough for our purpose, and we need a new geometric notion.

Definition 5.1. Let X and Y be two perfect derived stacks over a base field k . A correspondence from Y to X is a derived stack Z that is affine over $Y \times_k X$, i.e. $Z \in \mathrm{Aff}_{Y \times_k X}$ such that

$$\bullet (p_Y \circ \pi)_*(\mathcal{O}_Z) \simeq \mathcal{O}_Y,$$

- the composite of pushforward functors $\mathrm{QC}(Z) \xrightarrow{\pi_*} \mathrm{QC}(Y \times_k X) \xrightarrow{(p_Y)_*} \mathrm{QC}(Y)$ is conservative.

Here p_Y is the projection to Y , and $\pi : Z \rightarrow Y \times_k X$. Let $\mathrm{Cor}_k(Y, X)$ be the full subcategory of $(\mathrm{Aff}_{Y \times_k X})^\simeq$ spanned by correspondences from Y to X . We shall refer to $\mathrm{Cor}(Y, X)$ as the space (or ∞ -groupoid) of correspondences from Y to X .

Correspondences can be regarded as “twisted morphisms”. The notion of correspondences generalizes that of morphisms of derived stacks. Namely, the mapping space $\mathrm{Map}_{\mathrm{Sh}(\mathrm{Aff}_k)}(Y, X)$ is naturally embedded into $\mathrm{Cor}(Y, X)$ as a full subcategory, see Remark 5.4.

We define the composition of correspondences. Let X, Y , and Z are perfect derived stacks over k and $p_{YX} : Z \times_k Y \times_k X \rightarrow Y \times_k X$ the natural projection. The projections p_{ZY} and p_{YX} are defined in a similar manner. The projection p_{YX} induces $p_{YX}^* : \mathrm{Aff}_{Y \times_k X} \rightarrow \mathrm{Aff}_{Z \times_k Y \times_k X}$ given by the pullback $U \mapsto Z \times_k U$. Using $\mathrm{Aff}_{Y \times_k X} \simeq \mathrm{CAlg}(\mathrm{QC}(Y \times_k X))^{op}$ and $\mathrm{Aff}_{Z \times_k Y \times_k X} \simeq \mathrm{CAlg}(\mathrm{QC}(Z \times_k Y \times_k X))^{op}$, we define $(p_{YX})_* : \mathrm{Aff}_{Z \times_k Y \times_k X} \rightarrow \mathrm{Aff}_{Y \times_k X}$ by $W \mapsto (p_{YX})_*(W)$, where we regard W and $(p_{YX})_*(W)$ as objects in $\mathrm{CAlg}(\mathrm{QC}(Y \times_k X))$ and $\mathrm{CAlg}(\mathrm{QC}(Z \times_k Y \times_k X))$ respectively.

We define the map

$$\mathrm{Aff}_{Z \times_k Y} \times \mathrm{Aff}_{Y \times_k X} \rightarrow \mathrm{Aff}_{Z \times_k X}; \quad (V, U) \mapsto V \star U$$

by the formula $(V, U) \mapsto (p_{ZX})_*(p_{ZY}^*(V) \cdot p_{YX}^*(U))$. Here $p_{ZY}^*(V) \cdot p_{YX}^*(U)$ denotes the “intersection product” of $p_{ZY}^*(V)$ and $p_{YX}^*(U)$ on $Z \times_k Y \times_k X$, that is, $p_{ZY}^*(V) \cdot p_{YX}^*(U)$ is the fiber product of $p_{ZY}^*(V) \rightarrow Z \times_k Y \times_k X \leftarrow p_{YX}^*(U)$. As discussed in Remark 5.9, the composition $\mathrm{Aff}_{Z \times_k Y} \times \mathrm{Aff}_{Y \times_k X} \rightarrow \mathrm{Aff}_{Z \times_k X}$ induces

$$\mathrm{Cor}(Z, Y) \times \mathrm{Cor}(Y, X) \rightarrow \mathrm{Cor}(Z, X).$$

The diagonal $\Delta_X : X \rightarrow X \times_k X$ is the identity correspondence of X .

The purpose of this Section is to prove the following:

Theorem 5.2. *Let $X = [\mathrm{Spec} A/G]$ and $Y = [\mathrm{Spec} B/H]$ be two quotient stacks where $A, B \in \mathrm{CAlg}_k$, and G and H are pro-reductive groups over k .*

- (i) *There is a natural homotopy equivalence*

$$\mathrm{Cor}(Y, X) \rightarrow \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}_k^L)}(\mathrm{QC}^\otimes(X), \mathrm{QC}^\otimes(Y))$$

which carries Z to Z^ defined as*

$$Z^* : \mathrm{QC}^\otimes(X) \rightarrow \mathrm{QC}^\otimes(Y); \quad M \mapsto (p_Y)_*(p_X^*(M) \otimes_{\mathcal{O}_{Y \times_k X}} \mathcal{O}_Z)$$

where $p_X : Y \times_k X \rightarrow X$ and $p_Y : Y \times_k X \rightarrow Y$ are natural projections, and by $(-) \otimes_{\mathcal{O}_{Y \times_k X}} \mathcal{O}_Z$ we means the pullback along $\pi : Z \rightarrow Y \times_k X$.

- (ii) *Let $f : \mathrm{QC}^\otimes(X) \rightarrow \mathrm{QC}^\otimes(Y)$ and $g : \mathrm{QC}^\otimes(Y) \rightarrow \mathrm{QC}^\otimes(Z)$ be morphisms in $\mathrm{CAlg}(\mathrm{Pr}_k^L)$. Let $C_f \in \mathrm{Cor}(Y, X)$ and $C_g \in \mathrm{Cor}(Z, Y)$ be correspondences corresponding to f and g respectively. Then through the equivalence $\mathrm{Cor}(Z, X) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}_k^L)}(\mathrm{QC}^\otimes(X), \mathrm{QC}^\otimes(Z))$, the composite $g \circ f$ corresponds to $C_g \star C_f = (p_{ZX})_*(p_{ZY}^*(C_g) \cdot p_{YX}^*(C_f))$.*

Remark 5.3. For $Z \in \mathrm{Aff}_{Y \times_k X}$, the functor $Z^* : \mathrm{QC}(Y) \rightarrow \mathrm{QC}(X)$ given by

$$(p_X)_*(p_Y^*(-) \otimes_{\mathcal{O}_{Y \times_k X}} \mathcal{O}_Z)$$

is only a lax symmetric monoidal functor, but if we provide that $Z \in \mathrm{Cor}(Y, X)$, then Z is a symmetric monoidal functor (as the proof below indicates $(p_Y)_* \circ \pi_*$ is symmetric monoidal).

Remark 5.4. Let $f : Y \rightarrow X$ be a morphism of derived stacks. Then $(\mathrm{id}_Y, f) : Y \rightarrow Y \times_k X$ obtained from the identity and f is an affine morphism. Namely, $Y \rightarrow Y \times_k X \in \mathrm{Aff}_{Y \times_k X}$.

Indeed, for any $\phi : \operatorname{Spec} R \rightarrow Y$ and $\psi : \operatorname{Spec} R \rightarrow X$, if we denote by $\operatorname{Spec} R \times_{Y \times_k X} Y$ the fiber product of $\operatorname{Spec} R \xrightarrow{(\phi, \psi)} Y \times_k X \xleftarrow{(\operatorname{id}_Y, f)} Y$, then

$$\operatorname{Spec} R \times_{Y \times_k X} Y \simeq \operatorname{Spec} R \times_X \operatorname{Spec} R$$

where the right hand side is the fiber product associated to $\psi : \operatorname{Spec} R \rightarrow X$ and $f \circ \phi : \operatorname{Spec} R \rightarrow X$. It follows that (id_Y, f) is affine since X has affine diagonal. Hence we see that $(\operatorname{id}_Y, f) : Y \rightarrow Y \times_k X$ belongs to $\operatorname{Cor}(Y, X)$. It gives rise to a fully faithful functor

$$\operatorname{Map}_{\operatorname{Sh}(\operatorname{Aff}_k)}(Y, X) \rightarrow \operatorname{Cor}(Y, X).$$

Intuitively, we can think that this fully faithful functor carries $f : Y \rightarrow X$ to “the graph of f ”.

We need some Lemmata for the proof of Theorem 5.2.

The opposite ∞ -category $\operatorname{Cor}(Y, X)^{op}$ of correspondences can be naturally identified with the largest Kan subcomplex in the full subcategory of $\operatorname{CAlg}(\operatorname{QC}^\otimes(Y \times_k X)) \simeq \operatorname{CAlg}(\operatorname{QC}^\otimes(X) \otimes_k \operatorname{QC}^\otimes(Y))$ (cf. Section 2).

Let $\operatorname{CAlg}'(\operatorname{Pr}_k^L)_{\operatorname{QC}^\otimes(X \times_k Y)/}$ be the full subcategory of $\operatorname{CAlg}(\operatorname{Pr}_k^L)_{\operatorname{QC}^\otimes(X \times_k Y)/}$ spanned by those $\phi : \operatorname{QC}^\otimes(X \times_k Y) \rightarrow \mathcal{C}^\otimes$ such that $\operatorname{QC}^\otimes(Y) \xrightarrow{p_Y^*} \operatorname{QC}^\otimes(X \times_k Y) \xrightarrow{\phi} \mathcal{C}^\otimes$ is an equivalence.

There is a functor

$$\eta : \operatorname{Cor}(Y, X)^{op} \rightarrow \operatorname{CAlg}'(\operatorname{Pr}_k^L)_{\operatorname{QC}^\otimes(Y \times_k X)/}$$

which carries $\pi : W \rightarrow Y \times_k X$ to $\pi^* : \operatorname{QC}^\otimes(Y \times_k X) \rightarrow \operatorname{QC}^\otimes(W)$. More precisely, the functor η is given by the composite

$$\operatorname{Cor}(Y, X)^{op} \hookrightarrow \operatorname{Aff}_{Y \times_k X}^{op} \hookrightarrow (\operatorname{Sh}(\operatorname{Aff}_k)_{/Y \times_k X})^{op} \xrightarrow{\operatorname{QC}^\otimes} \operatorname{CAlg}'(\operatorname{Pr}_k^L)_{\operatorname{QC}^\otimes(Y \times_k X)/}$$

where two \hookrightarrow denote the natural inclusions. According to [31, 4.8.5.20] it is fully faithful. Moreover, we have:

Lemma 5.5. *The functor η induces an equivalence*

$$\operatorname{Cor}(Y, X)^{op} \rightarrow \operatorname{CAlg}'(\operatorname{Pr}_k^L)_{\operatorname{QC}^\otimes(Y \times_k X)/}.$$

Proof. We first show that for any $\pi : W \rightarrow Y \times_k X \in \operatorname{Cor}(Y, X) \subset \operatorname{Aff}_{Y \times_k X}$, $\eta(W)$ belongs to $\operatorname{CAlg}'(\operatorname{Pr}_k^L)_{\operatorname{QC}^\otimes(Y \times_k X)/}$. Namely, we will prove that $(p_Y \circ \pi)^* : \operatorname{QC}^\otimes(Y) \rightarrow \operatorname{QC}^\otimes(W)$ is an equivalence. Since $\pi : W \rightarrow Y \times_k X \in \operatorname{Cor}(Y, X)$, the pushforward $(p_Y \circ \pi)_* : \operatorname{QC}(W) \rightarrow \operatorname{QC}(Y)$ is conservative. Let $\{V_\lambda\}_{\lambda \in \Lambda}$ is a (small) set of compact (and dualizable) objects which generates $\operatorname{QC}(Y)$ as a stable presentable ∞ -category (notice that Y is perfect). Put $V'_\lambda = (p_Y \circ \pi)^*(V_\lambda)$. Observe that $\{V'_\lambda\}_{\lambda \in \Lambda}$ is a set of compact and dualizable objects which generates $\operatorname{QC}(W)$ as a stable presentable ∞ -category. Since $Y \times_k X$ and W are perfect, dualizable objects V'_λ are also compact in $\operatorname{QC}(W)$. Using the adjoint pair $(p_Y \circ \pi)^* : \operatorname{QC}(Y) \rightleftarrows \operatorname{QC}(W) : (p_Y \circ \pi)_*$ and the fact that $(p_Y \circ \pi)_*$ is conservative, we see that the vanishing $\operatorname{Hom}_{\operatorname{h}(\operatorname{QC}(W))}(V'_\lambda, N[r]) = 0$ for any $(\lambda, r) \in \Lambda \times \mathbb{Z}$ implies that $N \simeq 0$. By Proposition 2.1, $(p_Y \circ \pi)^* : \operatorname{QC}^\otimes(Y) \rightarrow \operatorname{QC}^\otimes(W)$ is extended to an equivalence $\operatorname{Mod}_{(p_Y \circ \pi)_*(\mathcal{O}_W)}^\otimes(\operatorname{QC}(Y)) \simeq \operatorname{QC}^\otimes(W)$ (the composite $\operatorname{QC}^\otimes(Y) \xrightarrow{(p_Y \circ \pi)_*(\mathcal{O}_W)^\otimes(-)} \operatorname{Mod}_{(p_Y \circ \pi)_*(\mathcal{O}_W)}^\otimes(\operatorname{QC}(Y)) \rightarrow \operatorname{QC}(W)$ is equivalent to $(p_Y \circ \pi)^*$). By the equivalence $(p_Y \circ \pi)_*(\mathcal{O}_W) \simeq \mathcal{O}_Y$, we see that $\operatorname{QC}^\otimes(Y) \simeq \operatorname{Mod}_{(p_Y \circ \pi)_*(\mathcal{O}_W)}^\otimes(\operatorname{QC}(Y)) \simeq \operatorname{QC}^\otimes(W)$.

Conversely, suppose that $\phi : \operatorname{QC}^\otimes(Y \times_k X) \rightarrow \mathcal{C}^\otimes$ belongs to $\operatorname{CAlg}'(\operatorname{Pr}_k^L)_{\operatorname{QC}^\otimes(Y \times_k X)/}$, that is, the composite $\phi \circ p_Y^* : \operatorname{QC}^\otimes(Y) \rightarrow \operatorname{QC}^\otimes(Y \times_k X) \rightarrow \mathcal{C}^\otimes$ is an equivalence. Let $\psi : \mathcal{C} \rightarrow \operatorname{QC}(Y \times_k X) \simeq \operatorname{QC}(Y \times_k X)$ be a (lax symmetric monoidal) right adjoint of ϕ . Put $A = \psi(1_{\mathcal{C}}) \in \operatorname{CAlg}(\operatorname{QC}(Y \times_k X))$ where $1_{\mathcal{C}}$ denotes the a unit of \mathcal{C} . The natural equivalence $\operatorname{Aff}_{Y \times_k X} \simeq \operatorname{CAlg}(\operatorname{QC}(Y \times_k X))$ (cf. Section 2) gives rise to $\pi : W \rightarrow Y \times_k X \in \operatorname{Aff}_{Y \times_k X}$. Moreover, $\operatorname{Mod}_A^\otimes(\operatorname{QC}(Y \times_k X)) \simeq \operatorname{QC}^\otimes(W)$ by Proposition 2.3. Since $\phi \circ p_Y^*$ is an equivalence and $Y \times_k X$ and Y are perfect, we can apply Proposition 2.1 to deduce that ϕ is extended to

$\text{Mod}_A^\otimes(\text{QC}(Y \times_k X)) \simeq \mathcal{C}^\otimes$. Therefore $\pi : W \rightarrow Y \times_k X$ lies in $\text{Cor}(Y, X)$, and we have the diagram

$$\begin{array}{ccc} & \text{QC}^\otimes(Y \times_k X) & \\ \pi^* \swarrow & \downarrow A \otimes (-) & \searrow \phi \\ \text{QC}^\otimes(W) & \xleftarrow{\simeq} \text{Mod}_A^\otimes(\text{QC}(Y \times_k X)) \xrightarrow{\simeq} & \mathcal{C}^\otimes. \end{array}$$

It follows that $\pi^* : \text{QC}^\otimes(Y \times_k X) \rightarrow \text{QC}^\otimes(W)$ is equivalent to ϕ in $\text{CAlg}'(\text{Pr}_k^L)_{\text{QC}^\otimes(Y \times_k X)/}^\simeq$. \square

Lemma 5.6. *There is a natural homotopy equivalence*

$$\text{Map}_{\text{CAlg}(\text{Pr}_k^L)}(\text{QC}^\otimes(X), \text{QC}^\otimes(Y)) \rightarrow \text{CAlg}'(\text{Pr}_k^L)_{\text{QC}^\otimes(Y \times_k X)/}^\simeq.$$

Proof. Let $\langle \text{id} : \text{QC}^\otimes(Y) \rightarrow \text{QC}^\otimes(Y) \rangle$ be the full subcategory of $\text{CAlg}(\text{Pr}_k^L)_{\text{QC}^\otimes(Y)/}^\simeq$ spanned by those objects $\text{QC}^\otimes(Y) \rightarrow \text{QC}^\otimes(Y)$ which is equivalent to the identity functor $\text{QC}^\otimes(Y) \rightarrow \text{QC}^\otimes(Y)$. It is obvious that $\langle \text{id} : \text{QC}^\otimes(Y) \rightarrow \text{QC}^\otimes(Y) \rangle$ is equivalent to a contractible space, i.e. Δ^0 . Note that if $\text{CAlg}''(\text{Pr}_k^L)_{\text{QC}^\otimes(Y \times_k X)/}^\simeq$ is the full subcategory of $\text{CAlg}'(\text{Pr}_k^L)_{\text{QC}^\otimes(Y \times_k X)/}^\simeq$ spanned by those objects $\phi : \text{QC}^\otimes(Y \times_k X) \rightarrow \mathcal{C}^\otimes$ such that $\mathcal{C}^\otimes = \text{QC}^\otimes(Y)$ and $\text{QC}^\otimes(Y) \xrightarrow{p_Y^*} \text{QC}^\otimes(Y \times_k X) \xrightarrow{\phi} \text{QC}^\otimes(Y)$ is equivalent to the identity. Then the inclusion

$$\text{CAlg}''(\text{Pr}_k^L)_{\text{QC}^\otimes(Y \times_k X)/}^\simeq \hookrightarrow \text{CAlg}'(\text{Pr}_k^L)_{\text{QC}^\otimes(Y \times_k X)/}^\simeq$$

is a homotopy equivalence. We have a pullback square

$$\begin{array}{ccc} \text{Map}_{\text{CAlg}(\text{Pr}_k^L)_{\text{QC}^\otimes(Y)/}^\simeq}(\text{QC}^\otimes(Y \times_k X), \text{QC}^\otimes(Y)) & \longrightarrow & \text{CAlg}''(\text{Pr}_k^L)_{\text{QC}^\otimes(Y \times_k X)/}^\simeq \\ \downarrow & & \downarrow \\ \langle \text{id} : \text{QC}^\otimes(Y) \rightarrow \text{QC}^\otimes(Y) \rangle & \longrightarrow & \text{CAlg}(\text{Pr}_k^L)_{\text{QC}^\otimes(Y)/}^\simeq \end{array}$$

in $\widehat{\mathcal{S}}$, where the right vertical functor is determined by $p_Y^* : \text{QC}^\otimes(Y) \rightarrow \text{QC}(Y \times_k X)$. The essential image of the right vertical functor is $\langle \text{id} : \text{QC}^\otimes(Y) \rightarrow \text{QC}^\otimes(Y) \rangle$, and the bottom horizontal arrow is a fully faithful functor. Therefore the top horizontal functor is an equivalence. By Proposition 4.6 $\text{QC}^\otimes(Y \times_k X) \simeq \text{QC}^\otimes(X) \otimes_k \text{QC}^\otimes(Y)$, and thus the adjoint pair

$$\text{QC}^\otimes(Y) \otimes_k (-) : \text{CAlg}(\text{Pr}_k^L) \rightleftarrows \text{CAlg}(\text{Mod}_{\text{QC}^\otimes(Y)}(\text{Pr}_k^L)) \simeq \text{CAlg}(\text{Pr}_k^L)_{\text{QC}^\otimes(Y)/} : \text{forget}$$

implies a homotopy equivalence

$$\text{Map}_{\text{CAlg}(\text{Pr}_k^L)_{\text{QC}^\otimes(Y)/}^\simeq}(\text{QC}^\otimes(Y \times_k X), \text{QC}^\otimes(Y)) \simeq \text{Map}_{\text{CAlg}(\text{Pr}_k^L)}(\text{QC}^\otimes(X), \text{QC}^\otimes(Y)).$$

Hence our assertion follows. \square

Proof of Theorem 5.2 (i). Our claim follows from Lemma 5.5 and Lemma 5.6. \square

Remark 5.7. Let $f : \text{QC}^\otimes(X) \rightarrow \text{QC}^\otimes(Y)$ be a morphism in $\text{CAlg}(\text{Pr}_k^L)$. The corresponding correspondence C_f is constructed as follows: Let $f_Y : \text{QC}^\otimes(X \times_k Y) \simeq \text{QC}^\otimes(X) \otimes_k \text{QC}^\otimes(Y) \rightarrow \text{QC}^\otimes(Y)$ be a morphism determined by f , the identity functor $\text{QC}^\otimes(Y) \rightarrow \text{QC}^\otimes(Y)$, and the universal property of the coproduct. Let f'_Y be a right adjoint of f_Y . Then as the proof of Lemma 5.5 reveals, C_f belonging to $\text{Cor}(Y, X) \subset \text{Aff}_{Y \times_k X} = \text{CAlg}(\text{QC}(Y \times_k X))^{op}$ is equivalent to $f'_Y(\mathcal{O}_Y)$ (note that f'_Y is a lax symmetric monoidal functor, and $f'_Y(\mathcal{O}_Y)$ lies in $\text{CAlg}(\text{QC}(Y \times_k X))$).

Remark 5.8. Suppose that X and Y are quasi-projective varieties over k . Then the above argument works also for X and Y , and we have an equivalence

$$\mathrm{Cor}(Y, X) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}_k^L)}(\mathrm{QC}^\otimes(X), \mathrm{QC}^\otimes(Y)).$$

It has been proved in [14] that $\mathrm{Cor}(Y, X)$ is naturally equivalent to $\mathrm{Map}_{\mathrm{Sh}(\mathrm{Aff}_k)}(Y, X)$. That is, every correspondence is a graph of a morphism.

Proof of Theorem 5.2 (ii). Identifying $\mathrm{QC}^\otimes(Z) \otimes_k \mathrm{QC}^\otimes(X)$ and $\mathrm{QC}^\otimes(Z) \otimes_k \mathrm{QC}^\otimes(Y)$ with $\mathrm{QC}^\otimes(Z \times_k X)$ and $\mathrm{QC}^\otimes(Z \times_k Y)$ respectively (Proposition 4.6), we have the diagram

$$\begin{array}{ccccc} \mathrm{QC}^\otimes(X) & \xrightarrow{f} & \mathrm{QC}^\otimes(Y) & & \\ p_X^* \downarrow & & \downarrow p_Y^* & \searrow g & \\ \mathrm{QC}^\otimes(Z) \otimes_k \mathrm{QC}^\otimes(X) & \xrightarrow{\mathrm{id}_Z \otimes f} & \mathrm{QC}^\otimes(Z) \otimes_k \mathrm{QC}^\otimes(Y) & \xrightarrow{g_Z} & \mathrm{QC}^\otimes(Z) \\ p_Z^* \uparrow & \nearrow & \nearrow \mathrm{id}_Z & & \\ \mathrm{QC}^\otimes(Z) & & & & \end{array}$$

where g_Z is determined by g , id_Z , and the universal property of the coproduct. Let A_f and A_g be the objects in $\mathrm{CAlg}(\mathrm{QC}(Y \times_k X))$ and $\mathrm{CAlg}(\mathrm{QC}(Z \times_k Y))$ that corresponds to C_f and C_g respectively. For a left adjoint functor F , we write F' for a right adjoint of F . Then by Remark 5.7, $A_g \simeq g'_Z(\mathcal{O}_Z)$. Note that $g_Z \circ (\mathrm{id}_Z \otimes f) \simeq (g \circ f)_Z$ where $(g \circ f)_Z$ is determined by $g \circ f$ and id_Z . Thus $(\mathrm{id}_Z \otimes f)'(A_g) \in \mathrm{CAlg}(\mathrm{QC}(X \times_k Z))$ corresponds to $g \circ f$. It will suffice to prove that $(\mathrm{id}_Z \otimes f)'(A_g)$ is equivalent to $(p_{ZX})_*(p_{ZY}^*(A_g) \otimes p_{YX}^*(A_f))$. To this end, unwind the construction of f obtained from A_f :

$$\mathrm{QC}^\otimes(X) \xrightarrow{p_X^*} \mathrm{QC}^\otimes(Y \times_k X) \xrightarrow{A_f \otimes (-)} \mathrm{Mod}_{A_f}^\otimes(\mathrm{QC}(Y \times_k X)) \xleftarrow{\sim} \mathrm{QC}^\otimes(Y).$$

Therefore, the right adjoint of $\mathrm{id}_Z \otimes f$ is the composite

$$\begin{aligned} \mathrm{QC}(Z) \otimes \mathrm{QC}(Y) &\xrightarrow{\sim} \mathrm{QC}(Z) \otimes \mathrm{Mod}_{A_f}(\mathrm{QC}(Y \times_k X)) && \simeq && \mathrm{QC}(Z \times_k C_f) \\ & && \xrightarrow{(\mathrm{id}_Z \times \pi)_*} && \mathrm{QC}(Z \times_k Y \times_k X) \\ & && \xrightarrow{(p_{ZX})_*} && \mathrm{QC}(Z \times_k X) \end{aligned}$$

where $\pi : C_f \rightarrow Y \times_k X$ denotes the structure morphism. The image of A_g under the composite is $(p_{ZX})_*(p_{ZY}^*(A_g) \otimes p_{YX}^*(A_f))$, as desired. \square

Remark 5.9. Theorem 5.2 (i) and (ii) implies that if $U \in \mathrm{Cor}(Y, X)$ and $V \in \mathrm{Cor}(Z, Y)$, $V \star U$ lies in $\mathrm{Cor}(Z, X)$. One can also prove it by verifying definition directly.

6. FINE ∞ -CATEGORIES AND EXAMPLES

In this Section, we give some examples and applications. For this purpose, we start with some usable results.

6.1. Elementary properties.

Proposition 6.1. *Let \mathcal{C}^\otimes be a symmetric monoidal idempotent complete additive category (the tensor product is additive separately in each variable). Suppose that the endomorphism algebra of a unit of \mathcal{C} is a field K of characteristic zero (hence \mathcal{C} is K -linear). Let C be a nonzero dualizable object in \mathcal{C} and suppose that the $(n+1)$ -fold wedge-product $\wedge^{n+1}C$ is a zero object. Suppose that n is the minimal natural number such that $\wedge^{n+1}C$ is a zero object. Then $\wedge^n C$ is invertible, i.e., $(\wedge^n C) \otimes (\wedge^n C)^\vee \simeq (\wedge^n C)^\vee \otimes (\wedge^n C)$ is a unit for some object $(\wedge^n C)^\vee$. In particular, C is a n -dimensional wedge-finite object.*

Proof. Let $\chi(C)$ be the trace defined as the element of $K := \mathrm{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$ given by

$$1_{\mathcal{C}} \rightarrow C^{\vee} \otimes C \xrightarrow{\mathrm{flip}} C \otimes C^{\vee} \rightarrow 1_{\mathcal{C}}$$

where the left map is the coevaluation and the right map is the evaluation. Taking account of $\wedge^{n+1}C \simeq 0$ we see by [23, Lemma 4.16, Corollary 4.20] that $\chi(C) = n \in \mathbb{Z} \subset K$. Since $\chi(\wedge^n C) = \frac{1}{n!} \chi(C)(\chi(C)-1) \cdots (\chi(C)-n+1)$, we have $\chi(\wedge^n C) = 1$. By combining $\chi(\wedge^n C) = 1$ and [23, Corollary 3.16, Corollary 4.20] $\wedge^2(\wedge^n C) \simeq 0$. Then according to [29, 8.2.9] $\wedge^n C$ is invertible. \square

Remark 6.2. In Proposition 6.1, if one drops the assumption on the endomorphism algebra of the unit, then the assertion does not hold. Namely, one can not deduce that C is wedge-finite from the condition that C is dualizable and $(n+1)$ -fold wedge-product $\wedge^{n+1}C$ is zero for some n . Let $X = \mathrm{Spec} A \sqcup \mathrm{Spec} B$ is a non-connected usual affine scheme and let L be an \mathcal{O}_X -module which is an invertible sheaf on $\mathrm{Spec} A$ and is zero on $\mathrm{Spec} B$. Then L is dualizable in the symmetric monoidal category of \mathcal{O}_X -modules and $\wedge^2 L \simeq 0$, but it is not an invertible object in the symmetric monoidal category of \mathcal{O}_X -modules.

Proposition 6.3. *Let \mathcal{X} be a sheaf $\mathrm{CAlg}_k \rightarrow \widehat{\mathcal{S}}$ such that $\mathrm{QC}^{\otimes}(\mathcal{X})$ is a fine ∞ -category. Let \mathcal{Y} be another sheaf and $f : \mathcal{Y} \rightarrow \mathcal{X}$ a relatively affine morphism, i.e., for any $\mathrm{Spec} A \rightarrow \mathcal{X}$ the fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathrm{Spec} A$ is affine. Then $\mathrm{QC}^{\otimes}(\mathcal{Y})$ is a fine ∞ -category.*

Proof. Let $\{V_{\lambda}\}_{\lambda \in \Lambda}$ be a set of wedge-finite objects such that $\{V_{\lambda}, V_{\lambda}^{\vee}\}_{\lambda \in \Lambda}$ generates $\mathrm{QC}^{\otimes}(\mathcal{X})$ as a symmetric monoidal stable presentable ∞ -category. Note that each wedge-finite object $p^*(V_{\lambda})$ is compact. Indeed, unwinding the definition of $\mathrm{QC}(\mathcal{X})$ and $\mathrm{QC}(\mathcal{Y})$ and using the base change formula [2, Section 3.2] we may assume that \mathcal{X} is affine. Then f_* preserves all small colimits and thus

$$\mathrm{Map}_{\mathrm{QC}(\mathcal{Y})}(f^*(V_{\lambda}), \varinjlim_i M_i) \simeq \mathrm{Map}_{\mathrm{QC}(\mathcal{X})}(V_{\lambda}, f_*(\varinjlim_i M_i)) \simeq \varinjlim_i \mathrm{Map}_{\mathrm{QC}(\mathcal{X})}(V_{\lambda}, f_*(M_i))$$

for any filtered colimit $\varinjlim_i M_i$. It follows also that the unit of $\mathrm{QC}^{\otimes}(\mathcal{Y})$ is compact. In addition, f_* is conservative, and by Remark 1.9 we see that the set $\{f^*(V_{\lambda}), f^*(V_{\lambda})^{\vee}\}_{\lambda \in \Lambda}$ of compact objects generates $\mathrm{QC}(\mathcal{Y})$ as a symmetric monoidal stable presentable ∞ -category. Hence $\mathrm{QC}^{\otimes}(\mathcal{Y})$ is fine. \square

Proposition 6.4. *Let \mathcal{C}^{\otimes} and \mathcal{D}^{\otimes} be two fine ∞ -categories. Then $\mathcal{C}^{\otimes} \otimes_k \mathcal{D}^{\otimes}$ is also fine.*

Proof. Combine Theorem 4.5 and Proposition 4.6. \square

6.2. Classical tannakian categories. We discuss a relationship with (classical) neutral tannakian categories. Let G be an algebraic group over a field k of characteristic zero. Let $\mathrm{QC}^{\otimes}(BG)$ be the k -linear stable presentable ∞ -category of quasi-coherent complexes over BG .

Let us observe that $\mathrm{QC}^{\otimes}(BG)$ is a fine ∞ -category. The symmetric monoidal ∞ -category $\mathrm{QC}(BG)$ is compactly generated, and compact and dualizable objects coincide (cf. [2, Corollary 3.22]). Take a closed immersion $G \hookrightarrow \mathrm{GL}_r$ that makes G a subgroup scheme of GL_r . Furthermore, by [43, Lemma 3.1] we can choose $G \hookrightarrow \mathrm{GL}_r$ so that GL_r/G is quasi-affine over k . The morphism $p : BG \rightarrow B\mathrm{GL}_r$ induced by $G \hookrightarrow \mathrm{GL}_r$ is quasi-affine since GL_r/G is a usual quasi-affine scheme (in particular, the structure sheaf is very ample). Let V be the standard representation of GL_r . Then by the standard use of the adjoint pair (see the proof of [2, Proposition 3.21]), the set $\{p^*(V), p^*(V)^{\vee}\}$ generates $\mathrm{QC}(BG)$ as a symmetric monoidal stable presentable ∞ -category. Note that $p^*(V)$ is compact and dualizable. Recall that V is wedge-finite, and so is $p^*(V)$. Therefore we conclude:

Proposition 6.5. *$\mathrm{QC}^{\otimes}(BG)$ is a fine algebraic ∞ -category.*

Corollary 6.6. *Let $[\mathrm{Spec} A/G]$ be a derived quotient stack, where an (possibly non-reductive) algebraic group G (over k) acts on $\mathrm{Spec} A$ with $A \in \mathrm{CAlg}_k$. Then $\mathrm{QC}^\otimes([\mathrm{Spec} A/G])$ is a fine algebraic ∞ -category.*

Proof. It follows from Proposition 6.3 and 6.5. \square

Remark 6.7. Let \mathcal{C}^\otimes be a k -linear symmetric monoidal stable presentable ∞ -category. By Corollary 6.6, the conditions in Theorem 4.1 are also equivalent to the condition: \mathcal{C}^\otimes is equivalent to $\mathrm{QC}^\otimes([\mathrm{Spec} A/G])$ for some $[\mathrm{Spec} A/G]$ such that an (possibly non-reductive) algebraic group G (over k) acts on $\mathrm{Spec} A$ with $A \in \mathrm{CAlg}_k$.

Remark 6.8. By Theorem 4.5, if G is a pro-reductive group, $\mathrm{QC}^\otimes(BG)$ is a fine ∞ -category. For an arbitrary pro-algebraic group G over k , $\mathrm{QC}^\otimes(BG)$ is not necessarily fine (since the unit is not compact when G has infinite cohomological dimension). For our purpose a correct generalization of $\mathrm{QC}^\otimes(BG)$ to arbitrary pro-algebraic groups is given by the Ind-category $\mathrm{Ind}^\otimes(\mathrm{Coh}(BG))$, where $\mathrm{Coh}(BG)$ is the stable subcategory of $\mathrm{QC}(BG)$ spanned by bounded complexes whose homology are finite dimensional. Namely, it is the symmetric monoidal compactly generated stable ∞ -category of *Ind-coherent complexes* on BG . For a pro-algebraic group G , $\mathrm{Ind}^\otimes(\mathrm{Coh}(BG))$ is a fine ∞ -category because the set of finite dimensional representations of G (that are wedge-finite) generates $\mathrm{Ind}^\otimes(\mathrm{Coh}(BG))$ as a stable presentable ∞ -category, and objects in $\mathrm{Coh}(BG)$ are compact in $\mathrm{Ind}^\otimes(\mathrm{Coh}(BG))$.

6.3. Stable ∞ -category of mixed motives, fine ∞ -categories and Kimura finiteness.

We study a relationship of fine ∞ -categories, the symmetric monoidal stable ∞ -category of mixed motives, and Kimura finiteness of Chow motives.

We begin by recalling briefly its background; why we should like to regard the category of *mixed motives* as a fine ∞ -category. One of the main themes of motives is motivic Galois theory which generalizes the classical Galois theory of fields (see e.g. [1]). The Galois theory for Artin motives corresponds to the classical Galois theory (cf. [1], [25, Section 8]). The Galois group of motives (motivic Galois group) should encode the structure of periods of motives. A conjectural abelian category of mixed motives is expected to be a tannakian category. Beside, it has been conjectured by Beilinson and Deligne that *the* abelian category of mixed motives should be the heart of a conjectural so-called motivic t -structure in the triangulated category of mixed motives DM that was constructed by Hanamura, Levine and Voevodsky. But the existence of a motivic t -structure is inaccessible by now (except the case of mixed Tate motives).

With this in mind, we study an ∞ -categorical enhancement of DM by means of fine ∞ -categories. That is, the above conjectural line and derived Tannakian viewpoint suggest the following picture. The ∞ -categorical enhancement of DM should constitute a fine ∞ -category, i.e., our ∞ -categorical analogue of tannakian category. A motivic Galois group should appear as the group object arising from the pointed derived stack corresponding to the fine ∞ -category equipped with a realization functor associated with a Weil cohomology theory. In [24], we have constructed derived automorphism group schemes of realization functors of mixed motives associated with a Weil cohomology theory (i.e. motivic Galois groups) by means of *tannakization* and also proved a consistency with the above traditional line (but, the construction of derived automorphism group schemes in [24] is somewhat abstract, whereas foundational properties are proved).

Now let us consider the \mathbb{Q} -linear stable presentable ∞ -category DM^\otimes of (Voevodsky's) mixed motives over a perfect field $S = \mathrm{Spec} K$, which is treated in [24], [25], [26], [35] (see these papers for details). We here use the symmetric monoidal model category DM^\otimes studied in [6, Example 7.15] and let DM^\otimes be the symmetric monoidal stable presentable ∞ -category (i.e., an object in $\mathrm{CAlg}(\mathrm{Pr}_S^L)$) obtained from (the full subcategory of cofibrant objects in) DM^\otimes by inverting weak equivalences. For a smooth variety X , i.e., a smooth scheme separated of finite type, there is a motif $M(X)$ of X that is an object of DM . We work with \mathbb{Q} -coefficients, that is, DM^\otimes is

constructed from chain complexes of Nisnevich sheaves of \mathbb{Q} -vector spaces with transfers on the category of finite correspondences [33], [6] (together with subsequent procedures: localizations with respect to homotopy invariance and descent, and an inversion of Tate objects). As a result, \mathbf{DM}^\otimes constitutes a \mathbb{Q} -linear symmetric monoidal presentable ∞ -category (see [24] for details). We can consider a direct generalization (of this subsection) to relative mixed motives over a smooth variety S , but for simplicity we consider the case when S is the Zariski spectrum of a perfect field.

Chow motives. There is a symmetric monoidal \mathbb{Q} -linear (ordinary) category CHM^\otimes of the (homological) Chow motives (cf. [38], see also [26, 4.1] for homological convention). In CHM , every object is dualizable. For a projective smooth variety X over K , there exist the Chow motif $h(X)$ in CHM and a symmetric monoidal \mathbb{Q} -linear fully faithful functor $CHM \rightarrow h(\mathbf{DM})$ which carries $h(X)$ to $M(X)$ (cf. [33, 20.2]). Hence Chow motives can be regarded as objects in \mathbf{DM} .

Kimura finiteness of Chow motives. The work of Kimura [28] and others places Kimura finiteness at the heart of recent developments of motivic theory. Let us recall this notion. An object M in CHM is evenly finite dimensional (resp. oddly finite dimensional) if there is a non-negative integer n such that $\wedge^n M = 0$ (resp. $\mathrm{Sym}^n M = 0$). Here $\mathrm{Sym}^n M$ denotes the symmetric product $\mathrm{Ker}(1 - \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma)$ where $\frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma : M^{\otimes n} \rightarrow M^{\otimes n}$ is the symmetrizer. An object M in CHM is Kimura finite dimensional if there exists a decomposition $M \simeq M^+ \oplus M^-$ such that M^+ is evenly finite dimensional and M^- is oddly finite dimensional. Similarly, we say that an object M in \mathbf{DM} is Kimura finite dimensional if there exists a decomposition $M \simeq M^+ \oplus M^-$ such that M^+ is evenly finite dimensional and M^- is oddly finite dimensional.

Lemma 6.9. *If M^+ is an evenly (resp. M^- is an oddly) finite dimensional object in \mathbf{DM} , then $M^+[2m]$ (resp. $M^-[2m+1]$) is wedge-finite for any $m \in \mathbb{Z}$. In particular, if M is a Kimura finite dimensional Chow motif such that $M \simeq M^+ \oplus M^-$ where M^+ is evenly finite dimensional and M^- is oddly finite dimensional, then $M^+[2m] \oplus M^-[2n+1]$ is wedge-finite for any $m, n \in \mathbb{Z}$.*

Proof. The endomorphism algebra of a unit of $h(\mathbf{DM})$ is \mathbb{Q} . Thanks to Proposition 6.1, M^+ is wedge-finite. By the Koszul sign rule (cf. [33, 8A.2]), $M^+[1]$ is oddly finite dimensional. Similarly, if M^+ is oddly finite dimensional, then $M^-[1]$ is evenly finite dimensional, and thus it is wedge-finite. Now our assertion is clear. \square

Let KF be a small set of objects in \mathbf{DM} that consists of Kimura finite dimensional Chow motives. (We remark that if M is Kimura finite dimensional, then the dual object M^\vee is Kimura finite dimensional.) Let \mathbf{DM}_{KF}^\otimes be a symmetric monoidal stable presentable full subcategory of \mathbf{DM} generated by $\{M\}_{M \in KF}$ as a symmetric monoidal stable presentable ∞ -category. Namely, it is the smallest stable full subcategory which contains the unit and $\{M\}_{M \in KF}$ and is closed under taking colimits, tensor products and duals. (We note that a dualizable object in \mathbf{DM}_{KF}^\otimes is not necessarily Kimura finite.) Known examples of Kimura finite objects are Chow motives $h(X)$ of abelian varieties (and more generally abelian schemes), some algebraic surfaces (rational surfaces, K3 surfaces of certain types, Godeaux surfaces..), Fano 3-folds, Tate objects $\mathbb{Q}(n)$ in \mathbf{DM} , and Artin motives, etc. We then have

Theorem 6.10. *The \mathbb{Q} -linear symmetric monoidal presentable ∞ -category \mathbf{DM}_{KF}^\otimes is a fine ∞ -category. Namely, there exist a derived stack $[\mathrm{Spec} A/G]$ where G is a pro-reductive group over \mathbb{Q} and an equivalence*

$$\mathbf{DM}_{KF}^\otimes \simeq \mathrm{QC}^\otimes([\mathrm{Spec} A/G]).$$

Proof. Note first that dualizable and compact objects coincide in \mathbf{DM} (see [7, Theorem 2.7.10]), and when X is a smooth projective variety, $M(X)$ is dualizable. Lemma 6.9 implies that \mathbf{DM}_{KF} admits a small set of wedge-finite objects which generates \mathbf{DM}_{KF} as a symmetric monoidal stable presentable ∞ -category (consider $M^+[2m] \oplus M^-[2n+1]$). Hence \mathbf{DM}_{KF} is a fine ∞ -category, and Theorem 4.5 implies our assertion. \square

Remark 6.11. Replacing KF by another set of objects one can consider many variants of Theorem 6.10.

Remark 6.12. It is important to notice that the existence of a motivic t -structure of DM_{KF}^\otimes is still unknown and mysterious, but Theorem 4.1 and 4.5 are applicable. For known cases of Kimura finiteness, Theorem 6.10 provides a unconditional application, which is a far-reaching generalization of the mixed Tate case. A precursor to the above Theorem in the case of mixed Tate motives is a theorem of Spitzweck (see [39]). The symmetric monoidal ∞ -category DM_{KF}^\otimes unconditionally contains the important class of mixed motives; mixed motives generated by abelian schemes (as a symmetric monoidal presentable ∞ -category).

The statement of the above form seems to be somewhat abstract. But, thanks to Proposition 4.8 and 4.12 we have the explicit presentation of A by means of motivic cohomology, Weyl construction and the product of general linear groups. We note that this presentation depends on the choice of a set of wedge-fine generators $\{C_\lambda\}_{\lambda \in \Lambda}$ that appears in Definition 1.3. For various applications (see the next Remark), it would be nice to have $\{C_\lambda\}_{\lambda \in \Lambda}$ such that each $R(C_\lambda)$ belongs to the heart of the standard t -structure of Mod_k (i.e., the concentrated in degree zero) where $R : \mathrm{DM}^\otimes \rightarrow \mathrm{Mod}_k^\otimes$ is a realization functor (e.g., étale, Betti, de Rham realizations). In all known Kimura finite cases at the writing of this paper, fortunately one can take such sets of wedge-finite generators.

Remark 6.13. Theorem 6.10 and variants can be applied to explicit constructions and studies of motivic Galois groups of DM_{KM}^\otimes by means of the construction of based loop spaces (equivariant bar construction) of $[\mathrm{Spec} A/G]$. Let

$$R : \mathrm{QC}^\otimes([\mathrm{Spec} A/G]) \simeq \mathrm{DM}_{KF}^\otimes \rightarrow \mathrm{Mod}_k^\otimes$$

be a realization functor associated to mixed Weil (co)homology with coefficients in k (see e.g. [24]). Suppose that each $R(C_\lambda)$ belongs to the heart of the standard t -structure of Mod_k for the prescribed set $\{C_\lambda\}$ of wedge-finite objects. Then as discussed in Remark 4.13 it gives rise to a morphism (“geometric point”)

$$p : \mathrm{Spec} k \rightarrow [\mathrm{Spec} A/G]$$

and the realization functor R can be identified with the pullback functor p^* . From this, we have the based loop space $\Omega_*[\mathrm{Spec} A/G] = \mathrm{Spec} k \times_{[\mathrm{Spec} A/G]} \mathrm{Spec} k$ that is a derived affine group scheme; similar constructions yield the Betti-de Rham comparison torsor, and motivic Galois group representing the automorphism group of the realization functor (see [25]). (This construction can be generalized to the context of realization of relative mixed motives.) The interested reader is referred to [26] and [25] for detailed study and further applications to mixed motives.

It is natural to expect

Conjecture 1. The \mathbb{Q} -linear symmetric monoidal stable presentable ∞ -category DM^\otimes is a fine ∞ -category.

Recall the following well-known conjecture:

Conjecture 2 (Kimura, O’Sullivan). Every object in CHM is Kimura finite dimensional.

The conjecture of Kimura and O’Sullivan does not imply the existence of a motivic t -structure on DM , but we have the following nice implication:

Proposition 6.14. *Conjecture 2 implies Conjecture 1.*

Proof. Note that by Conjecture 2 the set $\{M\}_{M \in CHM}$ of objects belonging to the essential image of $CHM \hookrightarrow \mathrm{h}(\mathrm{DM})$ generates DM as a stable presentable ∞ -category (cf. [7, 2.7.10]). Then (the proof of) Theorem 6.10 implies this Proposition. \square

Remark 6.15. Theorem 6.10 has a direct noncommutative variant. Let $\mathcal{M}_{loc}^{\otimes}$ be the symmetric monoidal stable ∞ -category of noncommutative mixed motives constructed (see e.g. [5]), which is a “universal domain” of localizing invariants. We suppose that $\mathcal{M}_{loc}^{\otimes}$ is \mathbb{Q} -linearized, that is, the base change to $\text{Mod}_{\mathbb{Q}}^{\otimes}$. The unit is compact. The stable full subcategory of $\mathcal{M}_{loc}^{\otimes}$ generated by Kimura finite objects in $\mathcal{M}_{loc}^{\otimes}$ as a symmetric monoidal stable presentable ∞ -category is a fine ∞ -category. The examples of Kimura finite objects in $\mathcal{M}_{loc}^{\otimes}$ is recently studied in [40].

6.4. Quasi-coherent complexes on an algebraic variety. We will apply our duality theorem to the derived ∞ -category of quasi-coherent sheaves on a quasi-projective variety. Let X be a quasi-projective scheme over a field k . Note that X admits a Zariski covering $\sqcup_{1 \leq i \leq n} \text{Spec } A_i \rightarrow X$ and its Čech nerve gives rise to a groupoid object $X_{\bullet} : N(\Delta)^{op} \rightarrow \text{Aff}_k$.

Let $\text{QC}^{\otimes}(X)$ be the k -linear symmetric monoidal ∞ -category of quasi-coherent complexes on X , that is, $\text{QC}^{\otimes}(X) := \varprojlim \text{QC}^{\otimes}(X_{\bullet}([n]))$. Let $\mathcal{D}_{qc}(X)$ be the derived ∞ -category of (ordinary) \mathcal{O}_X -modules whose cohomology is quasi-coherent on X (cf. [31, 1.3.5.8]). We then remark that there is an equivalence $\text{QC}(X) \simeq \mathcal{D}_{qc}(X)$ (indeed, by [32, VIII, 2.1.8, 2.3.1] there is an equivalence $\text{QC}(X)^+ \simeq \mathcal{D}_{qc}^+(X)$ between the full subcategories of left bounded objects with respect to the “standard” t -structures, and thus the left completeness of $\mathcal{D}_{qc}(X)$ and $\text{QC}(X)$ [17, B1], [32, VIII, 2.3.18] implies $\text{QC}(X) \simeq \mathcal{D}_{qc}(X)$).

Theorem 6.16. *Suppose that k is characteristic zero. The $\text{QC}^{\otimes}(X)$ is a fine ∞ -category and there exist a derived stack $[\text{Spec } A/\mathbb{G}_m]$ and an equivalence*

$$\text{QC}^{\otimes}(X) \simeq \text{QC}^{\otimes}([\text{Spec } A/\mathbb{G}_m])$$

where $\mathbb{G}_m = \text{GL}_1$. Moreover, there is an equivalence $A \simeq \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{\text{QC}(X)}(\mathcal{O}_X, \mathcal{L}^{\otimes r}) \otimes \chi_r$ in $\text{QC}(B\mathbb{G}_m)$ where χ_r is the character of weight r of \mathbb{G}_m , and \mathcal{L} is a very ample invertible sheaf.

Proof. Note first that $\text{QC}(X)$ is compactly generated, and dualizable and compact objects coincide (cf. [2]). Moreover, if \mathcal{L} is a very ample invertible sheaf on X a single compact object $\bigoplus_{0 \leq i \leq -d} \mathcal{L}^{\otimes i}$ for some $d \geq 0$ generates $\text{QC}(X)$ as a stable presentable ∞ -category (see [34, Theorem 4], [44, Lemma 3.2.2]). It follows that $\{\mathcal{L}, \mathcal{L}^{\vee}\}$ generates $\text{QC}^{\otimes}(X)$ as a symmetric monoidal stable presentable ∞ -category. Note that \mathcal{L}^{\vee} is wedge-finite and 1-dimensional. Let $A \otimes \chi_r$ denote the image of χ_r under the natural pullback functor $\text{QC}(B\mathbb{G}_m) \rightarrow \text{QC}([\text{Spec } A/\mathbb{G}_m])$. Then by Theorem 4.1 we obtain a derived stack $[\text{Spec } A/\mathbb{G}_m]$ and an equivalence $\text{QC}^{\otimes}([\text{Spec } A/\mathbb{G}_m]) \xrightarrow{\sim} \text{QC}^{\otimes}(X)$ in $\text{CAlg}(\text{Pr}_k^{\text{L}})$ which carries $A \otimes \chi_r$ to $\mathcal{L}^{\otimes(-r)}$. Therefore by Proposition 4.8 $A \simeq \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{\text{QC}(X)}(\mathcal{O}_X, \mathcal{L}^{\otimes r}) \otimes \chi_r$ in $\text{QC}(B\mathbb{G}_m)$, where $\text{Hom}_{\text{QC}(X)}(-, -)$ denote the hom complex. The truncation is given by $\pi_0(A) \simeq \bigoplus_{r \in \mathbb{Z}} H^0(X, \mathcal{L}^{\otimes r}) \otimes \chi_r$. \square

Remark 6.17. Actually, the assumption of characteristic zero on k is superfluous. When $\text{GL}_r = \mathbb{G}_m$, for the universal property of representations of GL_r discussed in Section 3, it is not necessary to assume that the coefficient field k is characteristic zero.

Recall Serre’s theorem which identifies the category of coherent sheaves on a projective variety X with the category of quasi-finitely generated graded modules of $\bigoplus_{r \in \mathbb{Z}} H^0(X, \mathcal{L}^{\otimes r})$ modulo torsion sheaves (see e.g. [18, Ex. 5.8]). We think of Theorem 6.16 as a *derived analogue of Serre’s theorem*. In spite of the equivalence $\text{QC}^{\otimes}(X) \simeq \text{QC}^{\otimes}([\text{Spec } A/\mathbb{G}_m])$, $[\text{Spec } A/\mathbb{G}_m]$ is not equivalent to X in general.

6.5. Coherent complexes on a topological space and Rational homotopy theory. We will discuss the ∞ -category of Ind-coherent complexes on a topological space from a viewpoint of rational homotopy theory. We work with coefficients in a fixed base field k of characteristic zero.

In his foundational work [15] Grothendieck developed the theory of Galois categories. The category $\text{Cov}(S)$ of finite topological covers of a topological space S is a Galois category. A base point s of S determines a symmetric monoidal functor $f : \text{Cov}(S) \rightarrow \text{Fin}$ to the category

of finite sets (with respect to cartesian monoidal structures), which carries a cover $\phi : X \rightarrow S$ to $\phi^{-1}(s)$. The automorphism group of f is equivalent to the pro-finite completion $\hat{\pi}_1(S, s)$ of the fundamental group $\pi_1(S, s)$. Moreover, $\hat{\pi}_1(S, s)$ continuously acts on the fiber $\phi^{-1}(s)$, and it gives rise to a categorical equivalence between $\text{Cov}(S)$ and the category of finite sets endowed with continuous $\hat{\pi}_1(S, s)$ -actions. We will describe a generalization of this story to rational homotopy theory by dint of fine ∞ -categories.

Let S be a connected topological space which we regard as an object in \mathcal{S} . We can think of S as a constant sheaf $\text{Aff}_k^{\text{op}} \rightarrow \mathcal{S}$ taking the value S . Let $\text{QC}^{\otimes}(S)$ denote the symmetric monoidal stable presentable ∞ -category of quasi-coherent complexes on S (cf. Section 2). If S is a contractible space, $\text{QC}^{\otimes}(S)$ is equivalent to Mod_k^{\otimes} . For an arbitrary (small) topological space S , $\text{QC}^{\otimes}(S)$ is the limit $\varprojlim_S \text{Mod}_k^{\otimes}$ of a constant diagram of Mod_k^{\otimes} indexed by the space S . The underlying ∞ -category $\text{QC}(S)$ is nothing but (equivalent to) the function complex $\text{Fun}(S, \text{Mod}_k)$. We will use the ∞ -category of Ind-coherent complexes of S instead of quasi-coherent complexes since dualizable objects on S are not necessarily compact objects. Let us define the full subcategory of bounded coherent complexes. Let $\text{Mod}_{k, \geq 0}$ (resp. $\text{Mod}_{k, \leq 0}$) be the full subcategory of Mod_k that consists of objects C such that $H_i(C) = 0$ for $i < 0$ (resp. $i > 0$). The pair $(\text{Mod}_{k, \geq 0}, \text{Mod}_{k, \leq 0})$ together with the truncation functors $\tau_{\geq 0} : \text{Mod}_k \rightarrow \text{Mod}_{k, \geq 0}$, $\tau_{\leq 0} : \text{Mod}_k \rightarrow \text{Mod}_{k, \leq 0}$ determines a t -structure on the stable ∞ -categories Mod_k . The pair $(\text{Fun}(S, \text{Mod}_{k, \geq 0}), \text{Fun}(S, \text{Mod}_{k, \leq 0}))$ determines a t -structure on $\text{Fun}(S, \text{Mod}_k)$. The composition with the truncation functors induces $\text{Fun}(S, \text{Mod}_k) \rightarrow \text{Fun}(S, \text{Mod}_{k, \geq 0})$ and $\text{Fun}(S, \text{Mod}_k) \rightarrow \text{Fun}(S, \text{Mod}_{k, \leq 0})$ which are a right adjoint functor and a left adjoint functor of $\text{Fun}(S, \text{Mod}_{k, \geq 0}) \subset \text{Fun}(S, \text{Mod}_k)$ and $\text{Fun}(S, \text{Mod}_{k, \leq 0}) \subset \text{Fun}(S, \text{Mod}_k)$ respectively. Let $\text{Coh}(S)$ be the full subcategory of $\text{QC}(S)$ that consists of those objects C such that C is bounded with respect to the t -structure and $H_i(C)$ is finite dimensional for every $i \in \mathbb{Z}$ (after pulling back to the heart of Mod_k). Namely, if $s : \Delta^0 \rightarrow S$ is a base point, then s^*C is represented by a bounded (chain) complex such that $H_i(s^*C)$ is finite dimensional for $i \in \mathbb{Z}$. (Notice that $\text{Coh}(S)$ can be defined to be the full subcategory of *dualizable objects*, i.e., perfect complexes.) The symmetric monoidal ∞ -category $\text{Coh}^{\otimes}(S)$ will play a role analogous to $\text{Cov}(S)$. We denote by $\text{ICoh}^{\otimes}(S) := \text{Ind}(\text{Coh}^{\otimes}(S))$ the symmetric monoidal stable presentable ∞ -category of Ind-objects and refer to it as the symmetric monoidal ∞ -category of Ind-coherent complexes on S . In this subsection, we observe that $\text{ICoh}^{\otimes}(S)$ is a fine ∞ -category and prove that the associated derived stack encodes the rational homotopy type of S under a certain finiteness condition.

Proposition 6.18. *Let S be a connected space. Then $\text{ICoh}^{\otimes}(S)$ is a fine ∞ -category.*

Let $G := \pi_1(S, s)$ be the fundamental group of S with respect to a fixed point $s \in S$. Let BG denote the fundamental groupoid of S and let $f : S \rightarrow BG$ be the natural projection.

Proof of Proposition 6.18. Observe first that the heart of $\text{Coh}(S)$ with respect to the t -structure is naturally equivalent to $\text{Fun}(S, \text{Vect}_k^f)$, where Vect_k^f is (the nerve of) the category of finite dimensional k -vector spaces regarded as the complexes placed in degree zero. Every functor $S \rightarrow \text{Vect}_k^f$ factors into $S \rightarrow BG \rightarrow \text{Vect}_k^f$ in a unique way. More precisely, we have a natural categorical equivalence $\text{Fun}(S, \text{Vect}_k^f) \simeq \text{Fun}(BG, \text{Vect}_k^f)$. Note that a functor $BG \rightarrow \text{Vect}_k^f$ amounts to an action of the group G on a finite dimensional vector space. Thus if G_{alg} denotes the pro-algebraic completion of G , then by the universal property of the completion, $\text{Fun}(BG, \text{Vect}_k^f)$ is equivalent to the category $\text{Vect}^f(G_{\text{alg}})$ of finite dimensional representations of G_{alg} as symmetric monoidal categories. Similarly, the heart of $\text{Coh}(BG)$ is equivalent to the category of finite dimensional representations of G_{alg} , and the pullback functor $f^* : \text{Coh}(BG) \rightarrow \text{Coh}(S)$ induces an identity of the heart when both hearts are identified with $\text{Vect}^f(G_{\text{alg}})$. Let G_{red} be the maximal pro-reductive quotient of G_{alg} , that is, the pro-reductive completion of G . We will prove that the set P of simple objects of $\text{Vect}^f(G_{\text{alg}})$ regarded as objects in the heart of $\text{Coh}(S)$ is a set of wedge-finite generator. Note that every object of $\text{Coh}(S)$ is compact in $\text{ICoh}(S)$, and every object of the heart of $\text{Coh}(S)$ is wedge-finite (since

wedge-finiteness can be verified after the base change along $s : \Delta^0 \rightarrow S$). Therefore, it is enough to show that $\mathrm{Coh}(S)$ is contained in the smallest stable subcategory \mathcal{C} which contains P . To see this, recall that every object of $\mathrm{Vect}^f(G_{\mathrm{alg}})$ has a filtration of finite length whose graded quotients are simple objects. Hence we find that the heart of $\mathrm{Coh}(S)$ is contained in \mathcal{C} . We then proceed by induction on the length with respect to t -structure. Suppose that objects D such that $H_i(D) = 0$ for $i < 0$ and $i > r$ belong to \mathcal{C} . Let C be an object in $\mathrm{Coh}(S)$. Assume that $H_i(C) = 0$ for $i < 0$ and $i > r + 1$. Then using the t -structure we have the distinguished triangle

$$\tau_{\geq 0}(C[-1])[1] \rightarrow C \rightarrow H_0(C) \rightarrow \tau_{\geq 0}(C[-1])[2].$$

By the assumption $\tau_{\geq 0}(C[-1])[1]$ belongs to \mathcal{C} . As observed above $H_0(C)$ belongs to \mathcal{C} . Thus C lies in \mathcal{C} . It follows that arbitrary shifts of C lie in \mathcal{C} , as desired. \square

Next we let P the set of simple objects that belongs to the part of degree zero in $\mathrm{Coh}(S)$ (cf. the proof of Proposition 6.18). An object in P can be thought of as a simple local system on S . We fix an order on the set of wedge-finite generator P . Using Proposition 6.18 and Theorem 4.5, we associate a derived quotient stack $\mathcal{X} := [\mathrm{Spec} A/H]$ to $\mathrm{ICoh}^\otimes(S)$ and the ordered set P where A is a commutative differential graded algebra that acts on H is a pro-reductive group and obtain an equivalence $\mathrm{QC}^\otimes(\mathcal{X}) \simeq \mathrm{ICoh}^\otimes(S)$ as objects in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$. We describe how the associated stack \mathcal{X} encodes the data of the rational homotopy type of S .

Remark 6.19. There are other choices of P . For example, we can choose the set of all (finite dimensional) semi-simple objects to be P . When G is a finite group, we can choose the set of a single faithful finite dimensional representation $\{V\}$ of G as P .

From the viewpoint of reconstruction problem, the initial categorical data should be a pair $(\mathrm{ICoh}^\otimes(S), s^* : \mathrm{ICoh}^\otimes(S) \rightarrow \mathrm{ICoh}^\otimes(\Delta^0) \simeq \mathrm{Mod}_k^\otimes)$ where we think of $\mathrm{ICoh}^\otimes(S)$ and s^* as an object in $\mathrm{CAlg}(\mathrm{Pr}_k^{\mathrm{L}})$ and a morphism respectively. The set P of simple local sytms can be obtained from the pair as follows. Let \mathcal{H}_k be the full subcategory of Mod_k spanned by those objects E such that $\mathrm{Hom}_{\mathrm{h}(\mathrm{Mod}_k)}(1_k, E)$ is finite dimensional, and $\mathrm{Hom}_{\mathrm{h}(\mathrm{Mod}_k)}(1_k, E[n]) = 0$ for $n \neq 0$ where 1_k is a unit of Mod_k^\otimes . Put $\mathcal{F} := (s^*)^{-1}(\mathcal{H}_k)$. Then P is the set of simple objects in the homotopy category of \mathcal{F} which is a k -linear category.

Let $\mathrm{Aut}(s^*) : \mathrm{Aff}_k^{\mathrm{op}} \rightarrow \mathrm{Grp}(\widehat{\mathcal{S}})$ be the automorphism group functor which carries $\mathrm{Spec} R$ to the “space of automorphism group” of the composite of symmetric monoidal functors

$$\mathrm{ICoh}^\otimes(S) \xrightarrow{s^*} \mathrm{ICoh}^\otimes(*) \simeq \mathrm{QC}^\otimes(*) \simeq \mathrm{Mod}_k^\otimes \xrightarrow{\otimes_k R} \mathrm{Mod}_R^\otimes$$

(see [24, Section 3] for the precise definition). Here s^* denotes the pullback along the point $s : \Delta^0 = * \rightarrow S$, and $\mathrm{Grp}(\widehat{\mathcal{S}})$ denotes the ∞ -category of group objects in $\widehat{\mathcal{S}}$. By the main result of [25] and the equivalence $\mathrm{ICoh}^\otimes(S) \simeq \mathrm{QC}^\otimes(\mathcal{X})$, $\mathrm{Aut}(s^*)$ is represented by the based loop stack $\mathrm{Spec} k \times_{\mathcal{X}} \mathrm{Spec} k = \Omega_* \mathcal{X}$, that is a derived affine group scheme (that is, a group object in Aff_k , see [24, Appendix]). Here let us recall how to get base points on stacks from s^* (Remark 4.13). The point of $\mathcal{X} = [\mathrm{Spec} A/H]$ is given by $w : [\mathrm{Spec} \Gamma(H)/H] \simeq \mathrm{Spec} k \rightarrow [\mathrm{Spec} A/H]$ where we identify the ring of functions $\Gamma(H)$ as the image of the unit of Mod_k under the right adjoint of the composite (i.e., the forgetful functor) $\mathrm{QC}^\otimes(BH) \rightarrow \mathrm{ICoh}^\otimes(S) \xrightarrow{s^*} \mathrm{QC}^\otimes(*) \simeq \mathrm{Mod}_k^\otimes$. Moreover, $w^* : \mathrm{QC}^\otimes([\mathrm{Spec} A/H]) \rightarrow \mathrm{QC}^\otimes(\mathrm{Spec} k) \simeq \mathrm{Mod}_k^\otimes$ can be naturally identified with $s^* : \mathrm{ICoh}^\otimes(S) \rightarrow \mathrm{Mod}_k^\otimes$. We here record the following result:

Proposition 6.20. *The automorphism group functor $\mathrm{Aut}(s^*) : \mathrm{Aff}_k^{\mathrm{op}} \rightarrow \mathrm{Grp}(\widehat{\mathcal{S}})$ is representable by the derived affine group scheme $\Omega_* \mathcal{X}$ over k .*

For a derived stack $\mathcal{Y} : \mathrm{CAlg}_k \rightarrow \mathcal{S}$ equipped with a base point $y : \mathrm{Spec} k \rightarrow \mathcal{Y}$ we denote by $\pi_i(\mathcal{Y}, y)$ the sheafification of the composite

$$\mathrm{CAlg}_k^{\mathrm{dis}} \hookrightarrow \mathrm{CAlg}_k \xrightarrow{(\mathcal{Y}, y)} \mathcal{S}_* \xrightarrow{\pi_i(-)} \mathrm{Grp}$$

(with respect to fpqc topology) where \mathbf{Grp} is the category of groups, $\mathcal{S}_* := \mathcal{S}_{\Delta^0/}$, and $\pi_i(-)$ is the i -th homotopy group with respect to the base point. $\mathbf{CAlg}_k^{\text{dis}}$ is the full subcategory of \mathbf{CAlg}_k spanned by discrete objects, i.e., those objects C such that $H_i(C) = 0$ for $i \neq 0$. It is equivalent to the nerve of ordinary commutative k -algebras.

Theorem 6.21. *Suppose that $\pi_1(S, s)$ is algebraically good (see below for this notion) and a universal cover of S has a homotopy type of a finite CW complex. Then we have*

$$\pi_i(\mathcal{X}, w) = \begin{cases} \pi_i(S, s)_{\text{uni}} & \text{for } i > 1 \\ \pi_1(S, s)_{\text{alg}} & \text{for } i = 1 \end{cases}$$

where $\pi_i(S, s)_{\text{uni}}$ is the pro-unipotent completion of $\pi_i(S, s)$.

Let $G \rightarrow G_{\text{alg}}(k)$ be the canonical homomorphism from the discrete group G to group of the k -valued points of G_{alg} . It gives rise to a morphism $BG \rightarrow BG_{\text{alg}}$, regarding BG as the constant functor, and we have the pullback functor $\mathbf{QC}^{\otimes}(BG_{\text{alg}}) \rightarrow \mathbf{QC}^{\otimes}(BG)$ and its restriction $\mathbf{Coh}^{\otimes}(BG_{\text{alg}}) \rightarrow \mathbf{Coh}^{\otimes}(BG)$. Here $\mathbf{Coh}^{\otimes}(BG_{\text{alg}})$ is defined in a similar way, i.e., it consists of bounded complexes with finite dimensional (co)homology. Consequently, $\mathbf{Coh}^{\otimes}(BG_{\text{alg}})$ coincides with the full subcategory spanned by dualizable (but not necessarily compact) objects. The full subcategory $\mathbf{Coh}(BG_{\text{red}})$ of $\mathbf{QC}(BG_{\text{red}})$ is defined in a similar way. In his important work [42] where the theory of affine stacks and schematizations of spaces are developed, Toën introduced the notion of algebraically goodness, which we will use. We shall recall this notion. Let $H^i(G_{\text{alg}}, -)$ (resp. $H^i(G, -)$) is the i -th derived functor of invariants $\mathbf{Vect}(G_{\text{alg}}) \rightarrow \mathbf{Vect}_k$, $V \mapsto V^{G_{\text{alg}}}$ (resp. $\mathbf{Vect}(G) \rightarrow \mathbf{Vect}_k$, $V \mapsto V^G$), where $\mathbf{Vect}(G)$ is the category of possibly infinite dimensional representations of the discrete group G , and \mathbf{Vect}_k is the category of k -vector spaces. The natural homomorphism $G \rightarrow G_{\text{alg}}(k)$ induces the natural transformation $H^i(G_{\text{alg}}, -) \rightarrow H^i(G, -)$. The group G is said to be algebraically good when the natural map $H^i(G_{\text{alg}}, V) \rightarrow H^i(G, V)$ is an isomorphism for every finite dimensional representation V of G_{alg} and every $i \in \mathbb{Z}$. As examples of algebraically groups, finite groups, finitely generated free group, finitely generated abelian groups, fundamental groups of Riemann surfaces are known. The proof of the next Lemma is routine and left to the reader.

Lemma 6.22. *Suppose that G is algebraically good. The natural functor $\mathbf{Coh}(BG_{\text{alg}}) \rightarrow \mathbf{Coh}(BG)$ is a categorical equivalence.*

Unwinding Theorem 4.5 and its proof including the inductive construction, we have the following additional properties of $[\text{Spec } A/H]$ and the equivalence $\mathbf{QC}^{\otimes}([\text{Spec } A/H]) \simeq \mathbf{ICoh}^{\otimes}(S)$. We can construct (i) a homomorphism $G_{\text{red}} \rightarrow H$, (ii) a symmetric monoidal k -linear functor $\mathbf{QC}(BH) \rightarrow \mathbf{QC}^{\otimes}([\text{Spec } A/H]) \simeq \mathbf{ICoh}^{\otimes}(S)$ which factors into

$$\mathbf{QC}(BH) \xrightarrow{t^*} \mathbf{QC}(BG_{\text{red}}) \rightarrow \mathbf{Ind}(\mathbf{Coh}(BG)) \xrightarrow{f^*} \mathbf{ICoh}(S),$$

where we abuse notation denoting by f^* the left Kan extension $\mathbf{Ind}(f^*)$ of the restriction $f^* : \mathbf{Coh}(BG) \rightarrow \mathbf{Coh}(S)$ (cf. [30, 5.3.5.10]) and t is the induced morphism $BG_{\text{red}} \rightarrow BH$.

Remark 6.23. There are some more remarks on the properties. The commutative algebra object A is the image of the unit 1_S of $\mathbf{ICoh}(S)$ under the lax symmetric monoidal right adjoint functor of $\mathbf{QC}(BH) \rightarrow \mathbf{ICoh}(S)$. The homomorphism $G_{\text{red}} \rightarrow H$ is a closed immersion. In particular, the induced morphism $BG_{\text{red}} \rightarrow BH$ is an affine morphism since H and G_{red} are pro-reductive. To see this, remember that the essential image of compact objects in $\mathbf{QC}(BH)$ under the constructed functor $t^* : \mathbf{QC}(BH) \rightarrow \mathbf{QC}(BG_{\text{red}})$ forms a set of compact generator of $\mathbf{QC}(BG_{\text{red}})$ (we have constructed such a functor). It follows that the right adjoint functor $t_* : \mathbf{QC}(BG_{\text{red}}) \rightarrow \mathbf{QC}(BH)$ is conservative. Suppose that the kernel G'_{red} of $G_{\text{red}} \rightarrow H$ is non-trivial. Take a non-trivial irreducible representation V of G'_{red} , and let $h_*V \in \mathbf{QC}(BG_{\text{red}})$ be the pushforward along the natural affine morphism $h : BG'_{\text{red}} \rightarrow BG_{\text{red}}$, that is not zero. But since the composite $BG'_{\text{red}} \rightarrow BH$ factors into $BG'_{\text{red}} \rightarrow \text{Spec } k \rightarrow BH$, thus $t_*h_*V \simeq 0$. It gives rise to a contradiction. We conclude that $G_{\text{red}} \rightarrow H$ is a closed immersion.

In the rest of this section, G is assumed to be algebraically good and a universal cover of S has a homotopy type of a finite CW complex.

Let $\pi : U \rightarrow S$ be a universal cover of S . Then we have a pullback diagram in \mathcal{S} ,

$$\begin{array}{ccc} U & \xrightarrow{\eta} & * \\ \pi \downarrow & & \downarrow q \\ S & \xrightarrow{f} & BG \end{array}$$

where $*$ denotes the contractible space. We also fix a base point s' of U lying over s . The natural morphism $f : S \rightarrow BG$ induces the adjunction $f^* : \mathrm{QC}(BG) \rightleftarrows \mathrm{QC}(S) : f_*$. Note that $S \times_{BG} * \simeq U$ is a simply connected finite CW complex. Therefore, by [3, Lemma 3.4, 3.17] η_* is conservative and preserves small colimits, and there is an equivalence $\mathrm{QC}^\otimes(U) \simeq \mathrm{Mod}_C^\otimes$ such that $C \in \mathrm{CAlg}_k$ is the pushforward η_* of the unit 1_U in $\mathrm{QC}^\otimes(U)$, and $\eta_* : \mathrm{QC}(U) \simeq \mathrm{Mod}_C \rightarrow \mathrm{Mod}_k \simeq \mathrm{QC}(*)$ is the forgetful functor. Applying the base change formula [2, Section 3.2] to the above square diagram we see that $q^* f_* 1_S$ is C , that is the singular cochain complex of U which belongs to $\mathrm{Coh}(*)$ (keep in mind that U is of finite type). Here 1_S denotes the unit of $\mathrm{QC}^\otimes(S)$. It follows that $f_* 1_S$ lies in $\mathrm{Coh}(BG)$ and the restriction $f^* : \mathrm{Coh}(BG) \rightleftarrows \mathrm{Coh}(S) : f_*$ is an adjunction. Take left Kan extensions $\mathrm{Ind}(f^*) : \mathrm{ICoh}(BG) \rightarrow \mathrm{ICoh}(S)$ and $\mathrm{Ind}(f_*) : \mathrm{ICoh}(S) \rightarrow \mathrm{ICoh}(BG)$ of $\mathrm{Coh}(BG) \rightarrow \mathrm{Coh}(S) \subset \mathrm{ICoh}(S)$ and $\mathrm{Coh}(S) \rightarrow \mathrm{Coh}(BG) \subset \mathrm{ICoh}(BG)$ respectively (cf. [30, 5.3.5.10]). It gives rise to an adjunction $f^* : \mathrm{ICoh}(BG) \rightleftarrows \mathrm{ICoh}(S) : f_*$ (we abuse notation by writing f^* and f_* for them). The natural morphism $g : BG_{alg} \rightarrow BG_{red}$ induced by the quotient map $G_{alg} \rightarrow G_{red}$ determines the pullback functor $g^* : \mathrm{Coh}(BG_{red}) \rightarrow \mathrm{Coh}(BG_{alg})$ and its left Kan extension $g_* : \mathrm{QC}(BG_{red}) \simeq \mathrm{Ind}(\mathrm{Coh}(BG_{red})) \rightarrow \mathrm{Ind}(\mathrm{Coh}(BG_{alg}))$ (we abuse notation). By adjoint functor theorem, we have a right adjoint functor which we denote by g_* . Therefore there are a sequence of adjunctions

$$\mathrm{QC}(BG_{red}) \simeq \mathrm{ICoh}(BG_{red}) \xrightleftharpoons[g_*]{g^*} \mathrm{ICoh}(BG_{alg}) \simeq \mathrm{ICoh}(BG) \xrightleftharpoons[f_*]{f^*} \mathrm{ICoh}(S).$$

The middle equivalence follows from Lemma 6.22. The right adjoint functors f_* and g_* are lax symmetric monoidal functors, and hence they carry commutative algebra objects to commutative algebra objects. We regard 1_S as an object of $\mathrm{CAlg}(\mathrm{ICoh}(S))$ and put $C = f_* 1_S \in \mathrm{CAlg}(\mathrm{Coh}(BG_{alg})) \simeq \mathrm{CAlg}(\mathrm{Coh}(BG))$ (strictly speaking, we abuse notation. The pullback of C to $\mathrm{Spec} k$ is $\eta_* 1_U$). Let $B \in \mathrm{CAlg}(\mathrm{QC}(BG_{red}))$ be $g_* f_* 1_S$. (Observe and keep in mind that since $C = f_* 1_S$ lies in $\mathrm{Coh}(BG_{alg})$, B coincides with the image of C under the right adjoint functor $\mathrm{QC}(BG_{alg}) \rightarrow \mathrm{QC}(BG_{red})$ of the pullback functor $\mathrm{QC}(BG_{red}) \rightarrow \mathrm{QC}(BG_{alg})$.) Recall the adjoint pair $t^* : \mathrm{QC}(BH) \rightleftarrows \mathrm{QC}(BG_{red}) : t_*$. The functor t_* sends B to A in $\mathrm{CAlg}(\mathrm{QC}(BH))$.

Lemma 6.24. *There exist natural equivalences of stacks*

$$[\mathrm{Spec} C/G_{alg}] \simeq [\mathrm{Spec} B/G_{red}] \simeq [\mathrm{Spec} A/H].$$

Proof. The statement is a slightly imprecise. We claim that there are equivalences of stacks regarded as restricted functors $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{dis}}, \widehat{\mathcal{S}})$. We will prove the first equivalence. It will suffice to show that the fiber product $\mathrm{Spec} k \times_{BG_{red}} [\mathrm{Spec} C/G_{alg}]$ is equivalent to $\mathrm{Spec} B$. $\mathrm{Spec} k \rightarrow BG_{red}$ is the natural projection. Notice first that the fiber product $BG_{alg} \times_{BG_{red}} \mathrm{Spec} k$ is equivalent to $[G_{red}/G_{alg}] \simeq BG_{uni}$ where G_{uni} is the unipotent radical of G_{alg} . Hence there is a fiber sequence of

$$\mathrm{Spec} C \rightarrow \mathrm{Spec} k \times_{BG_{red}} [\mathrm{Spec} C/G_{alg}] \simeq [\mathrm{Spec} C/G_{uni}] \rightarrow BG_{uni}$$

in $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{dis}}, \widehat{\mathcal{S}})$. By virtue of [42, 2.4.1], BG_{uni} can be represented by $\mathrm{Spec} E$ (as an object in $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{dis}}, \mathcal{S})$) such that E is a coconnective object in CAlg_k , i.e., $H_i(E) = 0$ for

$i > 0$. Moreover, since the sheafification of each homotopy group $[\mathrm{Spec} C/G_{uni}]$ is represented by a pro-unipotent group (by the fiber sequence and [42, 2.4.5]), we deduce from [42, 2.4.1] that $[\mathrm{Spec} C/G_{uni}] \simeq \mathrm{Spec} F$ such that F is a coconnective object that is the limit $\lim_{[n]} C \otimes_k \Gamma(G_{uni})^{\otimes n}$ (in CAlg_k or equivalently in the full subcategory of coconnective objects) of the cosimplicial diagram given by the Čech nerve associated to $\mathrm{Spec} C \rightarrow [\mathrm{Spec} C/G_{uni}]$ (the affinization [42, Section 2.2] commutes with colimits). Note that $F \simeq \lim_{[n]} C \otimes_k \Gamma(G_{uni})^{\otimes n}$ is the image of $C \in \mathrm{CAlg}(\mathrm{QC}(BG_{uni}))$ under the pushforward $\mathrm{QC}(BG_{uni}) \rightarrow \mathrm{QC}(\mathrm{Spec} k)$. Since C is homologically bounded above (as an object Mod_k), we can apply the base change formula for quasi-coherent complexes to the pullback diagram

$$\begin{array}{ccc} BG_{uni} & \longrightarrow & BG_{alg} \\ \downarrow & & \downarrow t \\ \mathrm{Spec} k & \longrightarrow & BG_{red} \end{array}$$

and conclude that $F \simeq B$. It yields a natural equivalence $[\mathrm{Spec} B/G_{red}] \simeq [\mathrm{Spec} C/G_{alg}]$. Since we have observed that $BG_{red} \rightarrow BH$ is an affine morphism with the fiber $H/G_{red} \simeq BG_{red} \times_{BH} \mathrm{Spec} k$ over the natural morphism $\mathrm{Spec} k \rightarrow BH$ (see Remark 6.23), the proof of $[\mathrm{Spec} B/G_{red}] \simeq [\mathrm{Spec} A/H]$ is similar and easier. \square

Proof of Theorem 6.21. As in the case of $[\mathrm{Spec} A/H]$ we obtain base points $u : \mathrm{Spec} k \rightarrow [\mathrm{Spec} C/G_{alg}]$ and $v : \mathrm{Spec} k \rightarrow [\mathrm{Spec} B/G_{red}]$ in a similar way. These points commute with the above equivalences $[\mathrm{Spec} C/G_{alg}] \simeq [\mathrm{Spec} B/G_{red}] \simeq [\mathrm{Spec} A/H]$ up to equivalence. Remember that the underlying complex of C is given by the chain complex computing the singular cohomology of U . Note that $\mathrm{QC}^{\otimes}(U) \simeq \mathrm{Mod}_C^{\otimes}$ (here C forgets the G_{alg} -action). If $D \simeq C \otimes_k \Gamma(G_{alg})$ denotes the image of the unit of Mod_C^{\otimes} under the pushforward along the composite $U \rightarrow S \rightarrow BG_{alg}$, we have a natural equivalence of stacks $\mathrm{Spec} C \simeq [\mathrm{Spec} D/G_{alg}]$. Thus the point $u : \mathrm{Spec} k \rightarrow [\mathrm{Spec} C/G_{alg}]$ factors into $\mathrm{Spec} k \simeq [\mathrm{Spec} \Gamma(G_{alg})/G_{alg}] \xrightarrow{u'} \mathrm{Spec} C \simeq [\mathrm{Spec} D/G_{alg}] \rightarrow [\mathrm{Spec} C/G_{alg}]$. According to [42, 2.3.3, 2.5.3], $\pi_i(\mathrm{Spec} C, u') \simeq \pi_i(U, s')_{uni}$. Combining the fiber sequence $\mathrm{Spec} C \rightarrow [\mathrm{Spec} C/G_{alg}] \rightarrow BG_{alg}$, the vanishing of the higher homotopy group of BG_{alg} and Lemma 6.24 we have isomorphisms

$$\pi_i([\mathrm{Spec} A/H], w) \simeq \pi_i([\mathrm{Spec} C/G_{alg}], u) \simeq \pi_i(\mathrm{Spec} C, u') \simeq \pi_i(U, s')_{uni}$$

for $i > 1$. For $i = 1$, $\pi_1([\mathrm{Spec} A/H], w) \simeq \pi_1([\mathrm{Spec} C/G_{alg}], u) \simeq G_{alg} = \pi_1(S, s)_{alg}$. This proves Theorem. \square

Remark 6.25. It is interesting to compare this subsection with a tannakian reconstruction of schemes and Deligne-Mumford stacks discussed in [14]. In *loc. cit.* emphasizing “derived tannakian viewpoint” we give a reconstruction of schemes and Deligne-Mumford stacks X from $\mathrm{QC}^{\otimes}(X)$ (without reference to any t -structure). Our approach to rational homotopy theory in this subsection gives a unified picture.

REFERENCES

- [1] Y. André, Une introduction aux motifs (motifs purs, motifs mixtes, periods), Panoramas et Synthèses, 17, Paris: Soc. Math. de France (2004).
- [2] D. Ben-Zvi, J. Francis and D. Nadler, Integral transforms and Drinfeld centers in derived algebraic geometry, J. Amer. Math. Soc. (2010) 909–966.
- [3] D. Ben-Zvi and D. Nadler, Loop spaces and connections, J. Topology 5 (2012) 377–430.
- [4] J. Bergner, A survey of $(\infty, 1)$ -categories, Towards Higher Categories, IMA Vol. in Math. and Its Applications, Springer (2010), 69–83.
- [5] A. Blumberg, D. Gepner and G. Tabuada, A universal characterization of higher algebraic K-theory, Geometry and Topology, 17 (2013) 733–838.
- [6] D.-C. Cisinski and F. Déglise, Local and stable homological algebra in Grothendieck abelian categories, Homotopy Homology Applications, 11(1), (2009), 219–260.
- [7] D.-C. Cisinski and F. Déglise, Mixed Weil cohomology, Adv. Math. 230 (2012) 55–130.

- [8] P. Deligne, Catégories tannakiennes, in The Grothendieck Festschrift, Vol. II, Progr. Math. 87, Birkhäuser Boston, MA, 1990.
- [9] P. Deligne, Catégories tensorielles, Mosc. Math. J. 2 (2002) 227–248. Issue dedicated to Y. Manin on the occasion of 65th birthday.
- [10] P. Deligne and J. S. Milne, Tannakian categories, Lecture Notes in Math. 900, Springer-Verlag (1982) 101–1061.
- [11] W. Dwyer and D. Kan, Function complexes in homotopical algebra, Topology 19 (1980) 427–440.
- [12] W. Fulton and J. Harris, Representation theory, A First Course, Springer (1991).
- [13] W. Fulton, Young Tableaux, London Math. Soc. Student Texts. (1997).
- [14] H. Fukuyama and I. Iwanari, Monoidal infinity category of complexes from Tannakian viewpoint, Math. Ann. 356, (2013), 519–553.
- [15] A. Grothendieck, Revêtements étales et Groupe Fondamental (SAG1), Lecture Notes in Math. 224, (1971), Springer-Verlag.
- [16] M. Groth, A short course on infinity-categories, arXiv:1007.2925
- [17] J. Hall, A. Neeman and D. Rydh, One positive and two negative results for derived category of algebraic stacks, preprint.
- [18] R. Hartshorne, Algebraic geometry, Springer.
- [19] V. Hinich, Homological algebra of homotopy algebras, Comm. in algebra 25 (1997), 3291–3323.
- [20] V. Hinich, Dwyer-Kan localization revisited, arXiv:1311.4128.
- [21] M. Hovey, Model categories, Math. surveys and Monographs 63, Amer. Math. Soc. (1999).
- [22] M. Hovey, Spectra and symmetric spectra in general model categories, J. Pure and App. Math. Vol 165, (2001), 63–127.
- [23] F. Ivorra, Finite dimensional motives and applications following Kimura, O’Sullivan and others, Autour des motifs, Asian-French summer school on algebraic geometry and number theory. Panoramas et synthèses.
- [24] I. Iwanari, Tannakization in derived algebraic geometry, J. K-Theory. 14 (2014), 642–700. available at my webpage <https://sites.google.com/site/isamuiwanarishomepage/>
- [25] I. Iwanari, Bar constructions and Tannakization, Publ. Res. Math. Sci. 50 (2014), 515–568.
- [26] I. Iwanari, Mixed motives and quotient stacks: Abelian varieties, preprint (2014), available at my webpage.
- [27] A. Joyal, Quasi-categories and Kan complexes, J. Pure Appl. Algebra 175 (2002), no. 1-3, 207222.
- [28] S. Kimura, Chow motives are finite dimensional, in some sense, Math. Ann. (2005).
- [29] S. Kimura, A note on finite dimensional motives, in “Algebraic Cycles and Motives” edited by J. Nagel and C. Peters, London Math. Soc. Lec. Notes 344 (2007), 203–213.
- [30] J. Lurie, Higher Topos Theory, Ann. Math. Studies, 170 (2009) Princeton Univ. Press.
- [31] J. Lurie, Higher Algebra, preprint, (2014). available at the author’s webpage.
- [32] J. Lurie, Derived Algebraic Geometry series, preprint available at the author’s webpage and arXiv
- [33] C. Mazza, V. Voevodsky and C. Weibel, Lecture Notes in Motivic Cohomology, Clay Math. Monograph vol.2 (2006).
- [34] D. Orlov, Remarks on generators and dimensions of triangulated category, arXiv:0804.1163.
- [35] M. Robalo, Noncommutative Motives I: A Universal Characterization of the Motivic Stable Homotopy Theory of Schemes, available at arXiv:1206.3645.
- [36] N. Saavedra Rivano, Categories Tannakiennes, Lecture Notes in Math. 265, Springer-Verlag (1972).
- [37] S. Schwede and B. Shipley, Stable model categories are categories of modules, Topology 42 (2003), 103–153.
- [38] T. Scholl, Classical motives, Motives (Seattle, WA, 1991), vol. I, Proc. Symp. Pure Math. vol. 55, Amer. Math. Soc., (1994), 163–187.
- [39] M. Spitzweck, Derived fundamental groups for Tate motives, arXiv:1005.2670
- [40] G. Tabuada, Voevodsky’s mixed motives versus Kontsevich’s noncommutative mixed motives, arXiv:1402.4438.
- [41] B. Toën and G. Vezzosi, Homotopical algebraic geometry I, II, Adv. in Math. 193 (2005), 257–372, Mem. Amer. math. Soc. no. 902 (2008)
- [42] B. Toën, Champs affines, Selecta Math. (N.S.), 12, (2006) 39–135.
- [43] B. Totaro, The resolution property for schemes and stacks, J. Reine Angew. Math. 577 (2004), 1–22.
- [44] M. Van den Bergh, Three-dimensional flops and noncommutative rings. Duke. Math. 122, (2004) 423–455.
- [45] V. Voevodsky, Triangulated category of motives, Chapter 5 of “Cycles, Transfers, and Motivic Homology Theories”, Ann. Math. Studies 143, (2000) Princeton Univ. Press.
- [46] J. Wallbridge, Tannaka duality for ring spectrum, preprint

MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, SENDAI, MIYAGI, 980-8578, JAPAN

E-mail address: iwanari@math.tohoku.ac.jp